Law II — Phase-bound Matter: Functorial Classification of Thermodynamic and Topological Phases

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Abstract

We present the second installment of the modular research series *Emergent Spacetime Dynamics*. Building *strictly modularly* on the categorical primitives established in Law I — the matter–information functor $M : \mathbf{Phys} \to \mathbf{Info}$, the symmetric monoidal structure of physical composition, and the sheaf semantics of local observables — we classify phases of matter as functorial equivalence classes. We first formalise the Landau paradigm as the connected-component invariant $\pi_0([\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}})$, where $\mathbf{Ham}_0$ is the groupoid of gapped local Hamiltonians under gap-preserving adiabatic equivalence and the superscript "eq" restricts to the on-site G-action. We then generalise beyond Landau by showing that topological order in (2+1)D corresponds to unitary modular tensor categories (MTCs), arising as Drinfeld centers $\mathcal{Z}(\mathcal{C})$ of unitary fusion categories. Symmetry-protected topological (SPT) phases are classified by group cohomology $H^{d+1}(G, U(1))$, identified with the homotopy classes of pointed maps $[\mathrm{B}G, \mathrm{B}^{d+1}U(1)]$. The topological entanglement entropy $\gamma = \log \mathcal{D}$ is constructed as the value of an entanglement-entropy functor on a connected disk; we show it is invariant under the equivalence relation of finite-depth local circuits. We provide worked computations for the toric code, the SSH chain, and the Levin–Wen string-net model, and we expose the precise composition hooks that Law III will consume in order to lift these classifications to the periodically driven setting. This paper is part of an explicitly *modular* framework: each law is a self-standing mathematical layer that composes functorially onto the prior laws, producing emergent properties at each level of composition. We do *not* claim a unified theory.

Contents

1 Introduction

1.1 Position in the modular series

This paper is Paper II of four in the modular research series *Emergent Spacetime Dynamics*. The series is organised as a sequence of categorical layers, each composing on the previous via an explicit functorial lifting:

$$\text{Law I} \xrightarrow{\text{Lift}_{\text{I}\to\text{II}}} \text{Law II} \xrightarrow{\text{Lift}_{\text{II}\to\text{III}}} \text{Law III} \xrightarrow{\text{Lift}_{\text{III}\to\text{IV}}} \text{Law IV}.$$

Law I (*Mathematical Formalisms*, MagnetonIO Research) established the categorical grammar that all subsequent laws share: symmetric monoidal categories, dagger-compact closed categories for quantum systems, sheaves and topoi for local observables, operads and factorisation algebras for QFT-style composition, the matter–information functor $M : \mathbf{Phys} \to \mathbf{Info}$, and a type-theoretic encoding of these structures. The present Paper II uses these primitives to classify phases of matter — both thermodynamic and topological — as functorial equivalence classes. We work entirely in the language of Law I.

We emphasise that the framework is *modular*, not unified. Each law contributes one self-standing mathematical layer; emergent properties at each layer arise from the *composition* of layers, not from any single overarching theory.

1.2 One-page recap of Law I

For convenience we collect the Law I primitives we will need, fixing notation and references for the remainder of the paper.

(L1.1) Symmetric monoidal category

A symmetric monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ consists of a category $\mathcal{C}$, a tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object I, and natural isomorphisms (associator α, unitors λ, ρ, symmetry σ) satisfying Mac Lane's pentagon and triangle axioms together with the symmetry hexagons. Mac Lane's coherence theorem guarantees that all rebracketings commute, so diagrams of structural isomorphisms can be elided in calculations.

(L1.2) Dagger-compact closed structure

A †-category is a category equipped with an involutive contravariant identity- on-objects functor $\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. A †-compact closed category additionally has duals $A \mapsto A^*$ with unit $\eta_A : I \to A^* \otimes A$ and counit $\varepsilon_A : A \otimes A^* \to I$. The category **FHilb** of finite-dimensional Hilbert spaces with linear maps is the canonical example. Traces, partial traces and the Born rule arise from (η, ε).

(L1.3) Matter–information functor

Law I posits a symmetric monoidal functor M : **Phys** $\to$ **Info** from a category of physical systems to a category of information-theoretic structures (sets of states, density matrices, classical channels, quantum channels). Whenever a physical category has a composition (tensor product), M preserves it. Concretely, the prototypical instance of M sends a quantum system with Hilbert space $\mathcal{H} \in$ **FHilb** to its space of density matrices $M(\mathcal{H}) = \{\rho \in \mathrm{End}(\mathcal{H}) : \rho \geq 0,\ \mathrm{Tr}\rho = 1\}$, regarded as the object of statistical states; on morphisms (unitary or completely positive maps) M acts by push-forward. We use M in the present paper to translate "Hilbert space of microstates" (an object of **Phys**) into "probability simplex of accessible macrostates" (an object of **Info**).

(L1.4) Sheaf semantics of local observables

Law I formalises local observables as a sheaf $\mathcal{O} : \mathrm{Open}(X)^{\mathrm{op}} \to$ **Alg** assigning algebras of operators to open regions. We use this to formulate locality conditions for order parameters and entanglement-entropy regions in this paper.

(L1.5) Functorial semantics

Following Lawvere, Law I treats algebraic theories as monoidal categories whose product-preserving functors into a base category are models. We instantiate this in section 3 by viewing a Landau theory as a functor $\mathrm{B}G \to$ **Ham**.

When we cite "(L1.k)" below we are using the Law I primitive of the same number.

1.3 Outline

The paper is organised as follows. In section 2 we define the category **Ham** of gapped Hamiltonians on a fixed lattice and introduce the equivalence relation of *gap-preserving adiabatic deformation*; phases are equivalence classes (i.e. connected components of **Ham**). We then define the functor pullback that classifies phases under symmetry. In section 3 we recover Landau's order-parameter paradigm in this language. In section 4 we treat SPT phases via group cohomology and identify the classifying functor with a generalised cohomology functor. Section 5 sets up unitary modular tensor categories as the data of a (2+1)D topological phase, with Kitaev's quantum double and the Levin–Wen construction as worked examples. Section 6 defines an entanglement-entropy functor and proves the

Kitaev–Preskill formula for $\gamma = \log \mathcal{D}$ in our setting. Section 8 works through three canonical examples: the toric code, the SSH chain, and the Fibonacci Levin–Wen model. Section 9 states the precise composition hooks Law III will consume. Section 12 lists open problems, and section 13 concludes.

2 Mathematical Framework: Phases as Equivalence Classes under Functor Pullback

2.1 The category of gapped Hamiltonians

Fix a (regular, finite or countable) lattice Λ and a finite-dimensional on-site Hilbert space $\mathfrak{h}$. Following Law I (L1.2), the total Hilbert space is the infinite tensor product (or a thermodynamic limit thereof) $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathfrak{h}$, an object of (a suitable completion of) **FHilb**.

Definition 2.1 (Local Hamiltonian)**.** A *local Hamiltonian* on Λ is a sum

$$H \;=\; \sum_{X\subset\Lambda} h_X, \qquad h_X \in \mathrm{End}(\mathcal{H}_X),$$

where each h_X is supported on a finite region $X \subset \Lambda$ with diameter bounded by a fixed range r, and the operator norm of h_X is uniformly bounded.

Definition 2.2 (Gap)**.** A local Hamiltonian H is *gapped* with gap $\Delta > 0$ if its spectrum on $\mathcal{H}_\Lambda$ has a single (possibly degenerate) ground subspace separated from the rest of the spectrum by at least Δ, uniformly as $|\Lambda| \to \infty$.

Definition 2.3 (Category **Ham**)**.** The category $\mathbf{Ham} = \mathbf{Ham}(\Lambda, \mathfrak{h})$ has:

- objects: gapped local Hamiltonians on Λ;
- morphisms $H_0 \to H_1$: gap-preserving piecewise-smooth paths $\{H_s\}_{s\in[0,1]}$ of local Hamiltonians with $H_{s=0} = H_0$, $H_{s=1} = H_1$, and $\inf_s \Delta(H_s) > 0$;
- composition: concatenation of paths;
- identity: the constant path.

By construction **Ham** is a groupoid (every path is reversible), and morphisms are homotopic iff they are connected through a homotopy of gap-preserving paths. We will write $\pi_0(\mathbf{Ham})$ for the set of connected components.

Remark 2.4. **Ham** is symmetric monoidal: the tensor product $H \boxtimes H'$ of two Hamiltonians on disjoint lattices is again a gapped local Hamiltonian on the union lattice; gap-preserving paths tensor componentwise. This is the direct image under the matter–information functor M (L1.3) of the symmetric monoidal structure on **Phys**.

2.2 Phases as functorial equivalence classes

Definition 2.5 (Phase)**.** A *phase* of matter (in the present setting) is a connected component of **Ham**:

$$\mathbf{Phase} \;=\; \pi_0(\mathbf{Ham}).$$

Two Hamiltonians belong to the same phase iff they are connected by a gap-preserving adiabatic path.

Remark 2.6. Three points deserve emphasis. (i) **Phase** is a set, not a category — the phase classification forgets the choice of path. (ii) The thermodynamic limit $|\Lambda| \to \infty$ is essential: at finite volume every two Hamiltonians can be connected by some path, but the gap typically closes uniformly only on a measure-zero subset. (iii) The lattice Λ and on-site Hilbert space $\mathfrak{h}$ are part of the data; phases are defined relative to a microscopic site structure.

Definition 2.7 (Symmetry action)**.** Let G be a group acting on $\mathfrak{h}$ by a unitary representation $\rho : G \to U(\mathfrak{h})$. The induced *on-site* action on $\mathcal{H}_\Lambda$ is $U_g = \bigotimes_{x\in\Lambda} \rho(g)$. We say $H \in \mathbf{Ham}$ is *G-symmetric* if $[U_g, H] = 0$ for all $g \in G$.

Definition 2.8 (Category $\mathbf{Ham}_G$)**.** The category $\mathbf{Ham}_G$ has G-symmetric Hamiltonians as objects, and gap-preserving paths $\{H_s\}$ *through G-symmetric Hamiltonians* as morphisms.

The forgetful functor $U : \mathbf{Ham}_G \to \mathbf{Ham}$ remembers only the underlying Hamiltonian without its G-equivariance. The induced map on connected components $U_* : \pi_0(\mathbf{Ham}_G) \to \pi_0(\mathbf{Ham})$ is in general neither injective nor surjective: phases that are distinct under G-symmetry can become equivalent when the symmetry is forgotten (this is precisely the SPT phenomenon, section 4).

2.3 The classifying space of G as a one-object category

For a group G, write $\mathrm{B}G$ for the one-object groupoid with morphism set G and composition given by group multiplication. A functor $\mathrm{B}G \to \mathcal{D}$ for any category $\mathcal{D}$ picks out an object D and a homomorphism $G \to \mathrm{Aut}_{\mathcal{D}}(D)$.

Proposition 2.9 (Symmetric Hamiltonians as $\mathrm{B}G$-functors)**.** *Let* $\mathbf{Ham}_0$ *denote the homotopy 1-category of* **Ham***: its objects are gapped local Hamiltonians on* $(\Lambda, \mathfrak{h})$*, and its morphisms are* homotopy classes *of gap-preserving paths. Explicitly,* $\mathrm{Hom}_{\mathbf{Ham}_0}(H, H')$ *is the set of path-homotopy classes* $[\{H_s\}_{s\in[0,1]}]$ *where* $H_{s=0} = H$*,* $H_{s=1} = H'$*, and* $\inf_s \Delta(H_s) > 0$*, under the equivalence relation that two paths are identified iff they are homotopic through gap-preserving paths fixing endpoints. Composition is concatenation of paths. With this definition* $\mathrm{Hom}_{\mathbf{Ham}_0}(H, H)$ *is a group (the fundamental group of the gapped locus at the basepoint* H*). Note that* $\mathbf{Ham}_0 = \pi_{\le 1}(\mathbf{Ham})$ *is the* 1*-truncation of the full* ∞*-groupoid* **Ham***; we will not need higher homotopy in this paper. Let* $[\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$ *denote the full subcategory of the functor category* $[\mathrm{B}G, \mathbf{Ham}_0]$ *consisting of those functors* F *for which the induced homomorphism* $G \to \mathrm{Aut}_{\mathbf{Ham}_0}(F(*))$ *factors through the homomorphism* $\rho_{\mathrm{ad}} : G \to \mathrm{Aut}_{\mathbf{Ham}_0}(F(*))$ *given by adjoint action* $\rho_{\mathrm{ad}}(g) = \mathrm{Ad}_{U_g}$ *where* $\mathrm{Ad}_{U_g}(H) = U_g H U_g^\dagger$*. (For G-symmetric Hamiltonians this adjoint action is the identity loop; the data of the functor is precisely a recording of which on-site action U_g is being used.) Then*

$$\mathbf{Ham}_G \;\simeq\; [\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}},$$

as groupoids.

Proof. We construct mutually inverse functors and check that they are equivalences.

Functor $\Phi : \mathbf{Ham}_G \to [\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$. Given a G-symmetric Hamiltonian H (so $[U_g, H] = 0$ for all $g \in G$), define a functor $F_H : \mathrm{B}G \to \mathbf{Ham}_0$ by setting $F_H(*) = H$ and, for each $g \in G$, $F_H(g) := [\mathrm{Ad}_{U_g}] \in \mathrm{Aut}_{\mathbf{Ham}_0}(H)$, the homotopy class of the loop $s \mapsto U_{g(s)} H U_{g(s)}^\dagger$ where $g(s)$ is any path in $U(\mathfrak{h})$ from $\mathbb{1}$ to the on-site representative U_g. Two observations make this well-defined and functorial. *(a)* Because $[U_g, H] = 0$ we have $U_g H U_g^\dagger = H$ on the nose, so the endpoint of the loop coincides with the basepoint H. *(b)* The on-site action gives a group homomorphism $G \to U(\mathfrak{h})^{\otimes\Lambda}$, so $\mathrm{Ad}_{U_{gh}} = \mathrm{Ad}_{U_g} \circ \mathrm{Ad}_{U_h}$ as automorphisms of $\mathbf{Ham}_0$; hence the assignment $g \mapsto [\mathrm{Ad}_{U_g}]$ is a homomorphism $G \to \mathrm{Aut}_{\mathbf{Ham}_0}(H)$, factoring through ρ_{ad} as required. This is the precise meaning of "factoring through ρ_{ad}." On morphisms (gap-preserving G-symmetric paths $\{H_s\}$), Φ sends $\{H_s\}$ to the natural transformation $\Phi(\{H_s\}) : F_{H_0} \Rightarrow F_{H_1}$ whose single component at $*$ is the homotopy class of the path itself. Naturality means the square

$$\begin{tikzcd}
H_0 \arrow[r, "{[\mathrm{Ad}_{U_g}]}"] \arrow[d, "\{H_s\}"'] & H_0 \arrow[d, "\{H_s\}"] \\
H_1 \arrow[r, "{[\mathrm{Ad}_{U_g}]}"] & H_1
\end{tikzcd}$$

commutes in $\mathbf{Ham}_0$ for every $g \in G$. This is exactly the condition that $\{H_s\}$ is a G-symmetric path, since both compositions equal the path $\{H_s\}$ itself — the top and bottom horizontal arrows are constant loops at the respective endpoints because each H_s commutes with U_g.

Functor $\Psi : [\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}} \to \mathbf{Ham}_G$. Given $F \in [\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$, set $\Psi(F) = F(*)$. By hypothesis $F(g) = [\mathrm{Ad}_{U_g}]$ in $\mathrm{Aut}_{\mathbf{Ham}_0}(F(*))$, i.e. the homotopy class of the adjoint loop $H_s = U_{g(s)} F(*) U_{g(s)}^\dagger$ for some choice of representative path. Since this loop is the constant loop in the homotopy category $\mathbf{Ham}_0$, we have $\mathrm{Ad}_{U_g}(F(*)) = U_g F(*) U_g^\dagger$ gap-equivalent to $F(*)$ for all $g \in G$. Combined with strict equality at the identity element $g = e$, the cocycle condition gives $U_g F(*) U_g^\dagger = F(*)$, i.e. $[U_g, F(*)] = 0$, so $F(*) \in \mathbf{Ham}_G$. On a natural transformation $F \Rightarrow F'$, Ψ takes its component at $*$.

Inverse equivalence. $\Psi \circ \Phi = \mathrm{id}_{\mathbf{Ham}_G}$ on the nose. For $\Phi \circ \Psi$, given F, the composite functor $F_{F(*)}$ has the same object value $F(*)$ and the same morphism values up to the on-site identification; the natural isomorphism $F \cong F_{F(*)}$ has identity component at $*$, so the two functors are equal up to canonical isomorphism. This yields a natural equivalence $\Phi \circ \Psi \cong \mathrm{id}$. □

Remark 2.10. The condition that the homomorphism $G \to \mathrm{Aut}_{\mathbf{Ham}_0}(F(*))$ factor through the on-site subgroup is essential. Concretely, the *on-site subgroup* of $\mathrm{Aut}_{\mathbf{Ham}_0}(H)$ is the image of $\rho_{\mathrm{ad}} : G \to \mathrm{Aut}_{\mathbf{Ham}_0}(H)$, which by definition consists of those automorphisms of H realised by adjoint action of unitaries of the form $\bigotimes_{x\in\Lambda} \rho(g)$ (with $\rho : G \to U(\mathfrak{h})$ a fixed on-site representation). An arbitrary functor $\mathrm{B}G \to \mathbf{Ham}_0$ would encode an arbitrary — not necessarily on-site — G-action on the Hamiltonian, and such functors classify projective representations and anomalies, which we treat in section 4 via group cohomology.

Remark 2.11. Theorem 2.9 is the formal mechanism by which Law I's functorial semantics (L1.5) classifies phases. The Landau paradigm is the special case where the target is the category of Hamiltonians built from a fixed local order-parameter degree of freedom; topological phases correspond to functors landing on Hamiltonians whose ground states are not products in any local basis.

2.4 Phases under symmetry as functor pullbacks

Definition 2.12 (Symmetric phase)**.** A *G-symmetric phase* is an element of $\pi_0(\mathbf{Ham}_G)$.

The map $U_* : \pi_0(\mathbf{Ham}_G) \to \pi_0(\mathbf{Ham})$ has a kernel and cokernel that encode, respectively, SPT phases (kernel) and spontaneous symmetry breaking (cokernel). This is one of the two main classification tools we use; the other is the assignment of algebraic invariants to ground states (modular data of an MTC, group-cohomology classes, etc.) which we develop in subsequent sections.

Theorem 2.13 (Phase-classification exact sequence)**.** *Let $\pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}} \subseteq \pi_0(\mathbf{Ham}_G)$ denote the* short-range entangled *G-symmetric phases, i.e. those phases whose ground state can be transformed into a product state by a finite-depth local quantum circuit that need not preserve G. Let $\pi_0(\mathbf{Ham})^{\mathrm{SRE}}$ be the corresponding set of (non-symmetric) SRE phases; this set has a single element, the trivial phase $*$, by Chen–Gu–Wen [5]. The forgetful map $U_* : \pi_0(\mathbf{Ham}_G) \to \pi_0(\mathbf{Ham})$ restricts to $U_*^{\mathrm{SRE}} : \pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}} \to \pi_0(\mathbf{Ham})^{\mathrm{SRE}} = \{*\}$, and the fibre $(U_*^{\mathrm{SRE}})^{-1}(*)$ is exactly the set of SPT phases* $\mathrm{SPT}^d(G)$ *defined in theorem 4.1. Concretely we have a sequence of pointed sets*

$$\mathrm{SPT}^d(G) \hookrightarrow \pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}} \xrightarrow{U_*^{\mathrm{SRE}}} \pi_0(\mathbf{Ham})^{\mathrm{SRE}} = \{*\},$$

which is exact in the sense that the kernel of U_*^{SRE} (the preimage of the basepoint $*$) is exactly the image of the inclusion. Adding spontaneously symmetry-broken phases extends the sequence: the cokernel of the inclusion $\mathrm{SPT}^d(G) \hookrightarrow \pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}}$ *classifies SSB patterns by subgroups $H \subseteq G$.*

Proof. Restriction to SRE phases is well-defined because the property of being SRE is preserved by gap-preserving deformations [5]. By definition of SRE, every SRE phase becomes the trivial phase upon forgetting symmetry, so U_*^{SRE} lands in $\{*\}$.

The inclusion $\mathrm{SPT}^d(G) \hookrightarrow \pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}}$ is by definition (theorem 4.1). Surjectivity onto the kernel: any $[H] \in \pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}}$ with $U_*[H] = *$ has, upon forgetting symmetry, a representative connected to a product state by a (non-symmetric) finite-depth circuit. By definition this is an SPT phase. Conversely, any SPT phase has SRE ground state (no anyons) and is in the kernel of U_* by construction.

For SSB patterns, take $[H] \in \pi_0(\mathbf{Ham}_G)$ whose ground state is a G-symmetric superposition of states each individually invariant only under a subgroup $H \subseteq G$. Modding out by SRE phases identifies $[H]$ with the data of the unbroken subgroup H; this is the standard Goldstone classification [11]. □

Remark 2.14. This theorem is best regarded as exhibiting an exact sequence of pointed sets in the homotopy category of phase classifications; we have not asserted higher homotopical content. A topos-theoretic refinement, in which one carries the full groupoid structure of $\mathbf{Ham}_G$ rather than just π_0, gives an analogous fibre sequence at the level of ∞-groupoids and is the natural setting for higher-dimensional generalisations. We do not pursue this here.

3 Landau Theory in Categorical Form

Before passing to topological phases we recover the Landau paradigm in the categorical language of section 2. This serves three purposes: it shows the present framework is at least as expressive as classical phase theory; it fixes notation that the topological theory generalises; and it isolates the precise feature of Landau theory — representability of an order parameter by a local section — that topological order violates.

3.1 Order parameters as local sections

Definition 3.1 (Order parameter representation)**.** Let G act on $\mathfrak{h}$. An *order-parameter representation* is a finite-dimensional unitary representation V of G together with a G-equivariant local map $\Phi_x : \mathrm{End}(\mathfrak{h}) \to V$ for each site $x \in \Lambda$, all related by translation.

In a state ω on $\mathcal{H}_\Lambda$, the order parameter is $\langle \Phi_x \rangle_\omega = \omega(\Phi_x) \in V$.

Definition 3.2 (Landau functor)**.** The *Landau functor* of a G-symmetric Hamiltonian H is

$$\mathcal{L}_H : \mathrm{B}G \to \mathbf{Vect}, \qquad \mathcal{L}_H(*) = V, \qquad \mathcal{L}_H(g) = \rho_V(g),$$

where V is the order-parameter representation and the assigned map records the action of G on the expectation $\langle \Phi_x \rangle$ in the ground state.

Proposition 3.3 (Landau classification, functorial form)**.** *Two G-symmetric Hamiltonians H_0, H_1 are in the same Landau phase iff their Landau functors $\mathcal{L}_{H_0}$ and $\mathcal{L}_{H_1}$ are naturally isomorphic as functors $\mathrm{B}G \to \mathbf{Vect}$. Equivalently, two Hamiltonians are in the same Landau phase iff their order-parameter expectations lie on the same G-orbit (up to V-isomorphism).*

Proof. A natural isomorphism $\eta : \mathcal{L}_{H_0} \Rightarrow \mathcal{L}_{H_1}$ is a G-intertwiner between the order-parameter representations. Existence of such an intertwiner is exactly the condition that the order parameters lie on the same G-orbit, which by Goldstone's theorem and standard Landau analysis is equivalent to lying in the same gap-preserving phase [11, 12]. □

Remark 3.4. The Ginzburg–Landau free energy $f[\varphi] = a(T)|\varphi|^2 + b|\varphi|^4 + c|\nabla\varphi|^2 + \cdots$ is recovered as the action of $\mathcal{L}_H$ on a generating loop in $\mathrm{B}G$, weighted by the relevant invariant polynomials in V.

3.2 Landau is not enough

The Landau classification is precisely the data $(V, \langle \Phi \rangle$ up to orbit). Two limitations are immediate.

(i) It is local: it sees only the value of an on-site (or short-range) functional.

(ii) It is symmetry-breaking-centric: it distinguishes phases via the residual subgroup $H \subseteq G$ stabilising $\langle \Phi \rangle$.

A topological phase is, by definition, one where $\langle \Phi \rangle \equiv 0$ for every local Φ and yet the ground subspace has structure (degeneracy, anyonic excitations, non-trivial entanglement) detectable only nonlocally. The Landau functor of a topological phase is trivial; we need a richer invariant.

4 Symmetry-Protected Topological Phases via Group Cohomology

4.1 SPT phases as the SSB-trivial subset

Definition 4.1 (SPT phase)**.** A *symmetry-protected topological (SPT) phase* of G-symmetric Hamiltonians is an element of $\ker U_* \subseteq \pi_0(\mathbf{Ham}_G)$: a G-symmetric phase that becomes trivial upon forgetting G-symmetry, but is nontrivial as a G-symmetric phase.

4.2 Group cohomology

For a group G and a G-module M, the standard bar complex computes group cohomology. The cochain in degree n is the abelian group $C^n(G, M) = \{f : G^n \to M\}$, with differential

$$\begin{aligned}(\delta f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ - \sum_{i=1}^{n} (-1)^{i-1} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ + (-1)^n f(g_1, \ldots, g_n).\end{aligned}$$

We write $Z^n = \ker \delta$, $B^n = \operatorname{im} \delta$, and $H^n(G, M) = Z^n/B^n$. We work with $M = U(1)$ throughout, with the trivial action.

4.3 The Chen–Gu–Liu–Wen classification

Theorem 4.2 (Chen–Gu–Liu–Wen [5])**.** *Let G be a finite group acting on-site, and let $d \geq 1$. The set of bosonic SPT phases on $\Lambda = \mathbb{Z}^d$ with on-site G-symmetry is in bijection with $H^{d+1}(G, U(1))$.*

The bijection is constructive: given a cocycle $\omega \in Z^{d+1}(G, U(1))$, one builds a fixed-point Hamiltonian whose ground state is a sum of group elements weighted by ω on simplices of a triangulation. We will not reproduce the full construction here; see [5]. The key categorical content for us is:

Proposition 4.3 (SPT classification as a functor)**.** *There is a functor*

$$\mathrm{SPT}^d : \mathbf{Grp} \to \mathbf{Ab}, \qquad G \mapsto H^{d+1}(G, U(1)),$$

and the assignment $H \mapsto [\omega_H]$ refines the phase classification map

$$\pi_0(\mathbf{Ham}_G) \;\to\; H^{d+1}(G, U(1))$$

on the SPT subset.

Proof. $H^{d+1}(-, U(1))$ is functorial in G via pullback along group homomorphisms; the map on phases is induced by restriction of the symmetry action. □

Example 4.4 (Haldane phase)**.** For $G = SO(3)$ and $d = 1$ (1D spin chain), one has

$$H^2(SO(3), U(1)) \cong \mathbb{Z}_2$$

[5]. Here $SO(3)$ is the rotational symmetry of an isotropic Heisenberg spin-1 chain, e.g. the AKLT chain

$$H_{\text{AKLT}} = \sum_i \Big[\vec{S}_i \cdot \vec{S}_{i+1} + \tfrac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \Big]. \tag{1}$$

Each site carries the spin-1 representation, and on-site $SO(3)$ acts diagonally on the chain. The non-trivial element of $H^2(SO(3), U(1))$ corresponds to the AKLT/Haldane phase. Concretely, the relevant 2-cocycle is the projective representation class of $SO(3)$ on the boundary spin-$\frac{1}{2}$ degree of freedom that emerges at each end of an open AKLT chain.

Example 4.5 (1D fermionic SPT)**.** The Kitaev chain (1D class D superconductor) is classified by $H^2(\mathbb{Z}_2^f, U(1)) \cong \mathbb{Z}_2$ in the bosonic-shadow framework, but the full fermionic classification uses spin cobordism rather than group cohomology [15]. We mention this only to flag that the classification $H^{d+1}(G, U(1))$ is the bosonic answer; fermionic systems require an upgrade to generalised cohomology, which is the open-problem item in section 12.

4.4 The classifying-space picture

The cohomology group $H^n(G, U(1))$ is the set of homotopy classes of pointed maps

$$H^n(G, U(1)) \;\cong\; [\mathrm{B}G, \mathrm{B}^n U(1)],$$

where $\mathrm{B}^n U(1) = K(U(1), n)$ is the Eilenberg–MacLane space. We will use this identification in section 9 to formulate the lifting to Floquet phases: Law III replaces $\mathrm{B}G$ with $\mathrm{B}G \times BS^1$ (adding the discrete time-translation group), and the relevant cohomology becomes $H^*(\mathrm{B}G \times BS^1, U(1))$, which by Künneth decomposes in a manner that exposes Floquet-specific phases.

5 Modular Tensor Categories and Topological Order

We now leave the Landau and SPT regimes and treat the most informationally rich phases: those with topological order.

5.1 Unitary fusion categories

Definition 5.1 (Unitary fusion category (UFC))**.** A UFC $\mathcal{C}$ is a $\mathbb{C}$-linear semisimple monoidal †-category with finitely many isomorphism classes of simple objects $\{X_i\}_{i \in I}$, a simple unit object $\mathbf{1} = X_0$, dual objects X_i^*, and unitary associators (F-matrices) satisfying the Mac Lane pentagon and unitarity.

We write $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ for the fusion multiplicity: $X_i \otimes X_j \cong \bigoplus_k N_{ij}^k X_k$.

5.2 Modular tensor categories

Definition 5.2 (MTC)**.** A *modular tensor category* is a UFC $\mathcal{C}$ equipped with a unitary braiding $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ and a ribbon twist $\theta_X : X \to X$ such that the modular S-matrix

$$S_{ij} = \frac{1}{\mathcal{D}} \mathrm{Tr}(\beta_{X_j,X_i} \circ \beta_{X_i,X_j}), \qquad \mathcal{D} = \sqrt{\sum_i d_i^2},$$

is non-degenerate, where d_i is the quantum dimension of X_i.

Remark 5.3. The non-degeneracy of S is the modular condition. By a theorem of Müger [16], S is non-degenerate iff the Müger center of $\mathcal{C}$ is trivial.

5.3 Topological order as MTC data

Theorem 5.4 (Wen, Levin–Wen, Kitaev: topological order is MTC data)**.** *Every* $(2{+}1)D$ *topologically ordered gapped phase admits, as ground-state algebraic data, a unitary modular tensor category* $\mathcal{C}$ *such that:*

(1) Simple objects of $\mathcal{C}$ *are anyon types.*

(2) Fusion rules are $X_i \otimes X_j \cong \bigoplus_k N_{ij}^k X_k$.

(3) Braiding statistics are encoded in β *and* θ.

(4) Ground-state degeneracy on a genus-g surface is $\mathrm{GSD}(g) = \sum_{i \in I} d_i^{2(g-1)}$ *(Verlinde formula [13]). At* $g = 1$ *this gives* $\mathrm{GSD}(1) = |I|$ *(number of simple objects, i.e. anyon types), valid for abelian* and *non-abelian theories. At higher genus the formula involves the quantum dimensions* d_i *explicitly.*

Conversely, the Levin–Wen string-net construction [6] produces, from a unitary fusion category $\mathcal{C}_0$*, a* $(2{+}1)D$ *commuting-projector lattice Hamiltonian whose anyon MTC is the Drinfeld center* $\mathcal{Z}(\mathcal{C}_0)$.

5.4 Kitaev's quantum double and the toric code

For a finite group G, the quantum double Hopf algebra is $D(G) = \mathrm{Fun}(G) \rtimes \mathbb{C}[G]$. Its representation category $\mathbf{Rep}(D(G))$ is a UMTC.

Example 5.5 (Toric code, on-site unitary $\mathbb{Z}_2$)**.** Here $G = \mathbb{Z}_2$ is the on-site *unitary* $\mathbb{Z}_2$ symmetry ("$\mathbb{Z}_2$ gauge" of the underlying lattice qubit; not a time-reversal or charge symmetry). The category $\mathbf{Rep}(D(\mathbb{Z}_2))$ has four simple objects: $\mathbf{1}$, e, m, $\varepsilon = em$, with fusion rules $e \otimes e = m \otimes m = \varepsilon \otimes \varepsilon = \mathbf{1}$, $e \otimes m = \varepsilon$, etc. All quantum dimensions are 1, so $\mathcal{D} = 2$ and $\gamma = \log 2$. The braiding of e around m is -1 (mutual semions), and ε has topological spin $\theta_\varepsilon = -1$ (a fermion).

The Kitaev Hamiltonian on the square lattice (qubit on each edge):

$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p, \qquad A_v = \prod_{e \ni v} \sigma_e^x, \qquad B_p = \prod_{e \in \partial p} \sigma_e^z, \tag{2}$$

realises a representative of the toric-code phase. All terms commute, and the ground space on a torus is four-dimensional, encoding the $[[n, 2, n^{1/2}]]$ stabiliser code that plays a central role in fault-tolerant quantum computation [1].

5.5 Levin–Wen string-nets and Drinfeld centers

Theorem 5.6 (Levin–Wen [6])**.** *Given a UFC $\mathcal{C}_0$ with simple objects $\{a_i\}$, F-symbols F^{abc}_{def}, and quantum dimensions $\{d_i\}$, define a lattice Hamiltonian on the honeycomb whose edges carry labels a_i and whose plaquette terms project onto the string-net condensate. The resulting model is exactly solvable, has gapped ground states, and the modular tensor category of its anyonic excitations is the Drinfeld center $\mathcal{Z}(\mathcal{C}_0)$.*

Example 5.7 (Doubled Fibonacci)**.** Let $\mathcal{C}_0 = \mathrm{Fib}$ be the Fibonacci fusion category (two simple objects $\mathbf{1}, \tau$ with $\tau \otimes \tau = \mathbf{1} \oplus \tau$, golden-ratio quantum dimension $d_\tau = \phi$). Then $\mathcal{Z}(\mathrm{Fib}) \cong \mathrm{Fib} \boxtimes \overline{\mathrm{Fib}}$. This is a non-abelian topological order capable of universal topological quantum computation [13].

5.6 Compositional connection to Law I

The MTC structure is exactly the Law I primitive of a braided monoidal category with duals, decorated by a non-degenerate S-matrix. The Drinfeld center $\mathcal{Z} : \mathbf{FusCat} \to \mathbf{MTC}$ is a functor on the Law I 2-category of fusion categories. We thus have:

$$\mathrm{Lift}^{\mathrm{top}}_{\mathrm{I}\to\mathrm{II}} : \ \mathbf{UFC}_{\mathrm{Law\,I}} \xrightarrow{\mathcal{Z}} \mathbf{MTC}_{\mathrm{Law\,II}} \xrightarrow{\text{phase}} \pi_0(\mathbf{Ham}). \tag{3}$$

This makes precise how the modular composition lifts Law I categorical data into Law II phase data.

6 Entanglement Entropy as a Functor

Topological entanglement entropy provides the primary observable bridge between MTC data and physical measurement. We formalise it as a functor in the spirit of Law I (L1.4).

6.1 Reduced density matrices and von Neumann entropy

For a state $|\psi\rangle \in \mathcal{H}_\Lambda = \mathcal{H}_A \otimes \mathcal{H}_B$ and a partition $\Lambda = A \sqcup B$, define $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ and $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$.

6.2 The entanglement-entropy functor

Definition 6.1 (Region category $\mathbf{Reg}_\Lambda$)**.** The objects of $\mathbf{Reg}_\Lambda$ are open subsets $A \subseteq \Lambda$; morphisms $A \to A'$ are inclusions $A \hookrightarrow A'$.

Definition 6.2 (Entanglement-entropy presheaf)**.** For a fixed pure state $|\psi\rangle \in \mathcal{H}_\Lambda$, the entanglement-entropy presheaf is the functor

$$S_\psi : \mathbf{Reg}^{\mathrm{op}}_\Lambda \to \mathbb{R}_{\geq 0}, \qquad A \mapsto S(A) = -\mathrm{Tr}(\rho_A \log \rho_A),$$

where the contravariance reflects that smaller regions sit inside larger ones, and the target $\mathbb{R}_{\geq 0}$ is regarded as a poset (order-preserving morphisms only).

By Law I (L1.4), S_ψ is a presheaf of (real, non-negative) numbers; it is emphatically *not* a sheaf in general — the gluing condition fails because entanglement is a global quantity that does not split additively under disjoint union.

6.3 The Kitaev–Preskill formula

Theorem 6.3 (Kitaev–Preskill, Levin–Wen [8])**.** *Let $|\psi\rangle$ be the ground state of a gapped, (2+1)D topologically ordered Hamiltonian with anyon MTC $\mathcal{C}$, and let $A \subset \Lambda$ be a topologically trivial disk-shaped region with smooth boundary of length L. Then*

$$S(A) = \alpha L - \gamma + O(e^{-L/\xi}), \tag{4}$$

where α is a non-universal constant proportional to L, ξ is the correlation length, and

$$\gamma = \log \mathcal{D}, \qquad \mathcal{D} = \sqrt{\sum_{i \in I} d_i^2}.$$

Proof idea. The Kitaev–Preskill construction uses the alternating sum $S_{KP} = -S(A_1) - S(A_2) - S(A_3) + S(A_1 \cup A_2) + S(A_2 \cup A_3) + S(A_1 \cup A_3) - S(A_1 \cup A_2 \cup A_3)$ for three regions A_1, A_2, A_3 meeting at a tri-junction. The boundary-length contributions cancel exactly, leaving the topological constant $S_{KP} = -\gamma = -\log \mathcal{D}$. Independently and contemporaneously, Levin and Wen [7] gave an alternative two-region construction extracting the same topological constant γ. □

Corollary 6.4 (γ is a phase invariant)**.** *γ is constant on each connected component of* **Ham** *(i.e. on each phase), because gap-preserving paths cannot change the MTC.*

Example 6.5 (Toric code TEE)**.** For the toric code, $\mathcal{D} = 2$ and $\gamma = \log 2$, in agreement with numerical [1] and analytical [8] results.

7 Renormalization Group as a Functor on Ham

We pause to make one further connection to Law I that will be needed by Laws III and IV: the renormalization group (RG) acting on **Ham**.

7.1 Block-spin RG as a coarsening functor

Fix a block-size $b \geq 2$ and group sites of Λ into blocks of b^d sites in spatial dimension d. Each block carries a Hilbert space $\mathfrak{h}^{\otimes b^d}$ of which we project onto a chosen subspace $\mathfrak{h}' \hookrightarrow \mathfrak{h}^{\otimes b^d}$. The induced map on Hamiltonians is the action of the block-spin RG transformation $R_b : \mathbf{Ham}(\Lambda, \mathfrak{h}) \to \mathbf{Ham}(\Lambda', \mathfrak{h}')$ where Λ' is the rescaled lattice.

Proposition 7.1 (RG functoriality)**.** *For each $b \geq 2$, R_b is a functor* **Ham** $\to$ **Ham**. *Its action on morphisms is defined by intertwining gap-preserving paths through the block-projector.*

Proof. We must check that gap-preserving paths $\{H_s\}$ map to gap-preserving paths $\{R_b H_s\}$. The block projector commutes with H_s at each s in the gapped fixed-point regime, so the gap of $R_b H_s$ is bounded below by the gap of H_s times the projector overlap, which is bounded uniformly along the path [12]. Composition $R_b \circ R_{b'} = R_{bb'}$ holds by associativity of block grouping. □

7.2 Fixed points and phase representatives

Definition 7.2 (RG fixed point)**.** $H \in \mathbf{Ham}$ is an *RG fixed point* (at scale b) if $R_b H = H$ up to a gap-preserving deformation, i.e. $[R_b H] = [H]$ in $\pi_0(\mathbf{Ham})$.

Example 7.3. The toric code Hamiltonian eq. (2) is an exact RG fixed point: the block projector for $b = 2$ on the toric code reproduces a toric code on the coarsened lattice.

Example 7.4. The Levin–Wen string-net Hamiltonian for any input UFC $\mathcal{C}_0$ is an exact RG fixed point. This is one of the principal motivations for the construction.

Remark 7.5. The set of RG fixed points up to gap-preserving deformation is a complete (and minimal) set of phase representatives: every phase contains at least one RG fixed point, and two fixed points in the same phase are related by a deformation. This justifies the practice of taking the toric code as the canonical representative of $\mathbb{Z}_2$ topological order.

7.3 Critical points and CFT

A non-fixed but RG-invariant point — a fixed point of the RG flow on the space of Hamiltonian densities — is a critical point. By the operator-product-expansion formalism [12], critical points are described by conformal field theories (CFTs); the data of a CFT (central charge, primary spectrum, OPE coefficients) is the algebraic content of the critical point. We will not develop CFT here, but we note that critical points are *not* objects of **Ham** (they are gapless), so they sit in the boundary $\partial\mathbf{Ham}$ of the gapped category. Phase boundaries are exactly such critical points.

8 Worked Examples

8.1 Example 1: The toric code

We summarise the analysis of theorem 5.5 and eq. (2) in our language:

- *Phase*: A topologically ordered phase, distinct from any Landau phase.
- *MTC*: $\mathbf{Rep}(D(\mathbb{Z}_2))$ with simple objects $\{\mathbf{1}, e, m, \varepsilon\}$, all quantum dimensions 1, $\mathcal{D} = 2$.
- *Functorial picture*: The functor $\mathrm{B}G \to \mathbf{Ham}$ for $G = \mathbb{Z}_2$ (the fermion-parity symmetry) lands on H_{TC}; the topological invariant is the isomorphism class of the resulting MTC, not the order parameter.
- *TEE*: $\gamma = \log 2$.
- *Code*: $[[n = 2L^2, k = 2, d = L]]$ on an $L \times L$ torus.

8.2 Example 2: The SSH chain (1D SPT)

The Su–Schrieffer–Heeger model is a 1D fermionic chain with alternating hopping amplitudes $t_1, t_2 > 0$:

$$H_{\text{SSH}} = \sum_j \Big(- t_1\, c^\dagger_{2j-1} c_{2j} - t_1\, c^\dagger_{2j} c_{2j-1} - t_2\, c^\dagger_{2j} c_{2j+1} - t_2\, c^\dagger_{2j+1} c_{2j} \Big) \tag{5}$$

For chiral symmetry $\Gamma = \sigma_z$, the model has two phases: $t_1 > t_2$ (trivial) and $t_1 < t_2$ (topological). The relevant classification is subtle because the SSH model is a *free-fermion* system, and the interplay of chiral symmetry, fermionic statistics and particle-hole/charge-conjugation structure pushes the classification beyond bosonic group cohomology and into the framework of K-theory and spin cobordism. *Bosonic-only* cohomology with on-site $\mathbb{Z}_2$-symmetry gives $H^2(\mathbb{Z}_2, U(1)) = 0$, so the SSH model is *not* a non-trivial bosonic SPT. The genuine non-triviality of the SSH model is fermionic: it is classified by spin cobordism, $\Omega^2_{\text{Spin}}(B\mathbb{Z}_2^\Gamma, U(1)) \cong \mathbb{Z}_2$ [15], with the non-trivial element matching the parity of the number of zero modes at the boundary. Equivalently, the chiral-symmetry-protected topological invariant is the winding number of the off-diagonal block of the Bloch Hamiltonian. We mention SSH here precisely to underline that the bosonic group-cohomology answer is *insufficient* for fermionic systems: this is the substance of open problem O1 in section 12.

In our setting, the SSH chain is a worked instance of $\mathbf{Ham}_G \to \pi_0$ with $G = \mathbb{Z}_2^\Gamma$ (chiral symmetry), and the topological invariant is the map

$$\pi_0(\mathbf{Ham}_{\mathbb{Z}_2^\Gamma})_{\text{SSH}} \;\to\; \mathbb{Z}_2, \quad H \mapsto \#\{\text{zero modes at boundary}\} \bmod 2.$$

8.3 Example 3: Doubled Fibonacci (Levin–Wen)

By theorem 5.6 and theorem 5.7, the Levin–Wen model with input $\mathcal{C}_0 = \text{Fib}$ is a non-abelian topologically ordered phase with anyon MTC $\mathcal{Z}(\text{Fib}) = \text{Fib} \boxtimes \overline{\text{Fib}}$. The total quantum dimension is

$$\mathcal{D} = (1+\phi^2)^{1/2} \cdot (1+\phi^2)^{1/2} = 1 + \phi^2 = 2 + \phi,$$

so $\gamma = \log(2+\phi) \approx 1.288$. The braid group representation on the fusion space of n Fibonacci anyons is dense in $SU(d_n)$, the basis of universal topological quantum computation [13].

8.4 Example 4: Explicit $\mathbb{Z}_2$ cocycles

To make theorem 4.3 concrete we compute $H^2(\mathbb{Z}_2, U(1))$ in two distinct module structures: *trivial* action (relevant for unitary on-site $\mathbb{Z}_2$ symmetry) and *anti-unitary* action $U_T(1)$ (relevant for time-reversal). We emphasise that *neither* is the Haldane-phase classifying group — the Haldane phase is protected by the on-site $SO(3)$ symmetry of the AKLT spin-1 chain, with $H^2(SO(3), U(1)) \cong \mathbb{Z}_2$ (theorem 4.4). The $\mathbb{Z}_2^T$ time-reversal SPT we compute below is a *different* $\mathbb{Z}_2$ class, realised e.g. in 1D Haldane-Dzyaloshinskii-Moriya chains protected by time reversal alone. We work multiplicatively (cochains take values in $U(1)$ written with multiplicative composition), which is the convention most natural for SPT applications.

The bar complex differential for an abelian-valued cochain on a group G with module action $g \cdot a$ is

$$(\delta^n \omega)(g_1, \ldots, g_{n+1}) = \Big(g_1 \cdot \omega(g_2, \ldots, g_{n+1})\Big) \cdot \prod_{i=1}^{n} \omega(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})^{(-1)^i} \cdot \omega(g_1, \ldots, g_n)^{(-1)^{n+1}}.$$

For a 1-cochain $f : G \to U(1)$ this gives

$$(\delta^1 f)(g_1, g_2) = (g_1 \cdot f(g_2)) \cdot f(g_1 g_2)^{-1} \cdot f(g_1). \tag{6}$$

For a 2-cochain $\omega : G^2 \to U(1)$ the cocycle condition $\delta^2 \omega = 1$ is

$$(g_1 \cdot \omega(g_2, g_3)) \cdot \omega(g_1 g_2, g_3)^{-1} \cdot \omega(g_1, g_2 g_3) \cdot \omega(g_1, g_2)^{-1} = 1. \tag{7}$$

Case 1: trivial action ($G = \mathbb{Z}_2 = \{e, g\}$, $g \cdot a = a$). Normalise $\omega(e, *) = \omega(*, e) = 1$ (always permissible by passing to the normalised bar complex). Setting $g_1 = g_2 = g_3 = g$ in eq. (7) and using $g^2 = e$ gives

$$\omega(g, g) \cdot \omega(e, g)^{-1} \cdot \omega(g, e) \cdot \omega(g, g)^{-1} = 1,$$

which is automatic from normalisation. So the only datum is $\omega(g, g) = z \in U(1)$, and *every* $z \in U(1)$ defines a normalised cocycle.

For coboundaries, with $f : \mathbb{Z}_2 \to U(1)$, $f(e) = 1$, $f(g) = t$, equation eq. (6) for trivial action gives

$$(\delta^1 f)(g, g) = f(g) \cdot f(g^2)^{-1} \cdot f(g) = t \cdot 1^{-1} \cdot t = t^2.$$

Hence the coboundaries are exactly the squares $\{t^2 : t \in U(1)\}$. Since $U(1)$ is divisible, every element is a square, so

$$H^2(\mathbb{Z}_2, U(1)) = U(1)/U(1)^2 = 0.$$

Case 2: anti-unitary action $U_T(1)$ ($G = \mathbb{Z}_2^T$, $g \cdot a = \bar{a} = a^{-1}$). Now $g \cdot f(g) = f(g)^{-1}$, so equation eq. (6) with $g_1 = g_2 = g$ gives

$$(\delta^1 f)(g, g) = (g \cdot f(g)) \cdot f(g^2)^{-1} \cdot f(g) = t^{-1} \cdot 1 \cdot t = 1.$$

Hence the only coboundary is the trivial cocycle. The cocycle condition still allows $\omega(g, g) = z \in U(1)$, but now we must check the action-twisted condition: substituting into eq. (7),

$$(g \cdot z) \cdot 1 \cdot z \cdot z^{-1} = z^{-1} \cdot z = 1,$$

which holds for all z. However, normalisation now requires $z \cdot \bar{z} = |z|^2 = 1$, so $z \in U(1)$. Modding out by coboundaries (which is now trivial) and by the further relation $z \sim z^{-1}$ (gauge equivalence under f with $f(g) = t$ and the modified action) gives exactly

$$H^2(\mathbb{Z}_2^T, U_T(1)) = \mathbb{Z}_2,$$

generated by the class $z = -1$. This is the time-reversal-protected SPT class for 1D spin chains, realised concretely by chains in which $T^2 = -1$ acts on a boundary spin-$\frac{1}{2}$ [5]. As noted at the head of this subsection, this is a *different* $\mathbb{Z}_2$ from that of the Haldane phase (theorem 4.4), which is classified by $H^2(SO(3), U(1))$ rather than by the $\mathbb{Z}_2^T$ cohomology computed here.

Why the trivial-action $\mathbb{Z}_2$ has no SPT in 1D. The vanishing $H^2(\mathbb{Z}_2, U(1)) = 0$ does *not* mean "no on-site $\mathbb{Z}_2$ SPT exists"; it means that with on-site *single-copy* $\mathbb{Z}_2$ symmetry alone there is no 1D SPT. The well-known cluster-state SPT in 1D is protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$, with non-trivial class in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ (one of the Künneth cross-terms). This illustrates the dependence of the SPT classification on the *choice* of symmetry group, even at fixed dimension.

8.5 Example 5: $\mathbb{Z}_2 \times \mathbb{Z}_2$ in 2D

In 2D, the relevant group is $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$. By Künneth and the standard computation,

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2^3,$$

giving eight distinct $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phases in 2D. Three of these come from the individual $\mathbb{Z}_2$ factors ($\mathbb{Z}_2^2$ from each copy of $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$), and the remaining factor of $\mathbb{Z}_2$ arises from the Künneth cross-term, corresponding to a mixed mutual-statistics SPT.

8.6 Example 6: Toric code as a stabiliser code

We expand the link between Law II and quantum information. The toric code Hamiltonian eq. (2) is the parent Hamiltonian of the stabiliser group

$$\mathcal{S} = \langle A_v, B_p : v \in V(\Lambda),\ p \in F(\Lambda) \rangle \subset \mathcal{P}_n$$

where $\mathcal{P}_n$ is the Pauli group on $n = 2L^2$ qubits (one per edge of an $L \times L$ torus). The stabiliser group satisfies $|\mathcal{S}| = 2^{n-2}$ (two relations $\prod_v A_v = \prod_p B_p = \mathbb{1}$), so the codespace has dimension $2^n/2^{n-2} = 4$, matching the four-fold degeneracy on the torus.

The two logical qubits are encoded by non-contractible Wilson loops:

$$\overline{X}_1 = \prod_{e \in C_1} \sigma_e^x, \quad \overline{Z}_1 = \prod_{e \in \widetilde{C}_1} \sigma_e^z, \quad \overline{X}_2, \overline{Z}_2 \text{ from the second non-contractible cycle.}$$

The code distance d equals the length of the shortest non-contractible loop, $d = L = \sqrt{n/2}$. Thus the toric code is a $[[2L^2, 2, L]]$ stabiliser code. This is the discrete realisation of the abstract data captured in the MTC $\mathbf{Rep}(D(\mathbb{Z}_2))$.

8.7 Example 7: Stacking and the abelian-group structure on SPT phases

SPT phases form an abelian group under stacking. Given two G-symmetric Hamiltonians H_1, H_2 on disjoint copies of the lattice, $H_1 \boxtimes H_2$ is a G-symmetric Hamiltonian on the union, with the diagonal G-action. The trivial phase is the unit, and the inverse of a layer is its time reversal.

By theorem 4.3, the stacking map matches cohomology addition:

$$[\omega_{H_1 \boxtimes H_2}] \ = \ [\omega_{H_1}] + [\omega_{H_2}],$$

both classes lying in $H^{d+1}(G, U(1))$. This is the categorical statement that "stacking SPTs adds their cocycles."

9 Composition Hooks for Law III

We close by stating the precise data Law III is to consume in lifting Law II to the periodically driven setting. The reader of Law III should treat this section as an interface specification.

9.1 Hook 1: The category Ham extends to $\mathbf{Ham}_{T\text{-per}}$

Define $\mathbf{Ham}_{T\text{-per}}$ as the category whose objects are T-periodic paths $H : S^1_T \to \mathbf{Ham}$ landing on gapped Hamiltonians and whose morphisms are gap-preserving homotopies through T-periodic families. The static embedding $\iota : \mathbf{Ham} \hookrightarrow \mathbf{Ham}_{T\text{-per}}$ sends H to the constant path.

9.2 Hook 2: Symmetry group enlarges by $\mathbb{Z}_T$

The on-site symmetry G in Law II is enlarged in Law III by the discrete time-translation symmetry $\mathbb{Z}_T$ generated by the period operator. The relevant classifying space becomes $\mathrm{B}G \times B\mathbb{Z}_T$, and the SPT-style classification is by $H^{d+2}(G \times \mathbb{Z}_T, U(1))$ in spatial dimension d (the extra degree arises from the S^1 factor in spacetime).

9.3 Hook 3: MTC enriches by $\mathbf{Rep}(\mathbb{Z}_T)$

The MTC of an equilibrium topological phase is $\mathcal{C}$. The Floquet enrichment is

$$\mathcal{C}_T = \mathcal{C} \boxtimes \mathbf{Rep}(\mathbb{Z}_T),$$

encoding micromotion-twisted sectors. Anomalous Floquet phases (Rudner–Lindner type) correspond to MTCs in this enrichment that are not products of an equilibrium MTC with $\mathbf{Rep}(\mathbb{Z}_T)$.

9.4 Hook 4: TEE extends to Floquet TEE

The Floquet entanglement-entropy functor

$$S_T : \mathbf{Reg}_\Lambda^{\mathrm{op}} \times \mathbb{Z}_T \to \mathbb{R}_{\geq 0}, \qquad (A, n) \mapsto S(A; \text{at stroboscopic time } nT),$$

has a topological constant $\gamma_T = \log \mathcal{D}_T$ where $\mathcal{D}_T$ is the total quantum dimension of $\mathcal{C}_T$.

9.5 Hook 5: Composition functor

The lifting from Law II to Law III is the functor

$$\mathrm{Lift}_{\mathrm{II}\to\mathrm{III}} : [\mathrm{B}G, \mathbf{Ham}] \to [\mathrm{B}(G \times \mathbb{Z}_T), \mathbf{Ham}_{T\text{-per}}], \qquad F \mapsto \mathrm{Fl}(F),$$

where $\mathrm{Fl}(F)$ is the Floquet-enriched functor. Time-crystalline phases are detected by natural transformations $\eta : \mathrm{Fl}(F) \Rightarrow \mathrm{Fl}^{\mathbb{Z}_2}(F)$ that are not isomorphisms (i.e. that detect period doubling).

10 Results

We summarise the principal results of this paper.

R1. *Phases as connected components.* Phases of matter on a fixed lattice are elements of $\pi_0(\mathbf{Ham})$, the connected components of the groupoid of gapped local Hamiltonians under gap-preserving adiabatic equivalence (theorem 2.5).

R2. *Symmetric phases as functor-pullback classes.* The category of G-symmetric Hamiltonians is equivalent to the functor category $[\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$, so symmetric phases are functorial equivalence classes (theorem 2.9).

R3. *Landau as functorial semantics.* The Landau classification is the natural-isomorphism classification of order-parameter functors $\mathcal{L}_H : \mathrm{B}G \to \mathbf{Vect}$ (theorem 3.3).

R4. *SPT as group cohomology, functorially.* The classification map

$$\pi_0(\mathbf{Ham}_G) \;\to\; H^{d+1}(G, U(1))$$

refines to a functor $\mathrm{SPT}^d : \mathbf{Grp} \to \mathbf{Ab}$ (theorem 4.3).

R5. *Topological order is MTC data.* A (2+1)D topologically ordered phase is classified by a unitary modular tensor category, arising as the Drinfeld center of a fusion category (Levin–Wen), with the data lifted from Law I via the functor $\mathcal{Z}$ (eq. (3)).

R6. *TEE as a phase invariant.* Topological entanglement entropy $\gamma = \log \mathcal{D}$ is constant on each phase (theorem 6.4) and is computed by the Kitaev–Preskill formula (theorem 6.3).

R7. *Composition hooks.* We have stated five precise hooks (section 9) by which Law III may consume Law II to produce Floquet classifications.

11 Discussion

11.1 Why functorial classification?

The choice to classify phases as functorial equivalence classes is not aesthetic. It is forced by the modular composition principle of the series: each law must produce data that the next law can consume *as a category*, not merely as a set or group. The set $\pi_0(\mathbf{Ham})$ alone is too coarse to support the lifting to Floquet phases (Law III) or to information geometry (Law IV); we need at minimum the category $\mathbf{Ham}_G$ and the fusion/braiding data of MTCs to be available as categorical structure that the next law's functor can act on.

11.2 What this paper does *not* claim

This paper does *not* claim:

(i) A complete classification of (3+1)D topological phases (this is open; cf. open problem 3 in section 12).

(ii) A categorical framework for fracton order (this is genuinely open; the string-net construction does not directly apply because of the sub-extensive ground-state degeneracy).

(iii) A non-perturbative bridge to quantum gravity (this is the territory of Law IV).

We are explicit that the modular framework is not a unified theory and that the classification at the topological-order level is the most rigorous tier; the lifting to Law III and beyond carries increasing speculative content.

11.3 Relation to other categorical frameworks

Our setup is closest to that of [6, 13] for (2+1)D and to [5] for SPT phases. The cobordism-classification programme of [15] extends the SPT classification to fermionic and anti-unitary symmetries; the bosonic group-cohomology classification we use here is the simpler sub-case. Our explicit functorial reformulation in terms of the Law I primitives $M : \mathbf{Phys} \to \mathbf{Info}$ and the sheaf semantics is the new contribution; the underlying mathematical results are due to the cited authors.

12 Open Problems

We list, in order of decreasing rigor of currently available techniques, open problems that bear directly on the modular composition.

O1. *Fermionic SPT classification beyond bosonic shadow.* Extend theorem 4.3 to a functor on the category of finite groups equipped with a $\mathbb{Z}_2^f$-extension, taking values in spin-cobordism cohomology $\Omega^*_{\mathrm{Spin}}(\mathrm{B}G, U(1))$.

O2. *(3+1)D topological order.* Is every (3+1)D topological phase the boundary of a (4+1)D Walker–Wang model with input fusion 2-category? A categorical answer would generalise theorem 5.6 from (2+1)D to (3+1)D.

O3. *Fracton order.* Find a categorical framework that captures the sub-extensive ground-state degeneracy and immobile excitations of fracton phases. The string-net construction does *not* apply.

O4. *Functorial RG.* Make precise Wilson's RG as a functor (or 2-functor) on **Ham**. Are RG fixed points exactly the rigid objects?

O5. *Non-invertible categorical symmetries.* Generalise the SPT classification of theorem 4.2 from groups G to fusion categories $\mathcal{S}$ acting as symmetries; the relevant cohomology is the second cohomology of $\mathcal{S}$ in $U(1)$.

O6. *Fibration of phases over the parameter space.* Make precise the fibration Phases $\to$ Couplings whose fibres are connected components of **Ham**, and study its monodromy (this connects to Law IV's information geometry).

13 Conclusion

We have provided a functorial classification of phases of matter, building strictly on the categorical primitives of Law I and producing in clean form the data Law III will consume. The principal conceptual point is that phases are equivalence classes of functors, not just states; the Landau paradigm and its topological generalisations both fit into this scheme, with the difference being the target category (Vect for Landau, Ham for SPT, MTC for topologically ordered phases). The topological entanglement entropy $\gamma = \log \mathcal{D}$ provides the principal numerical bridge between the algebraic data (the MTC) and the physical state (the ground subspace).

The composition with Law III is by replacing the symmetry group G with $G \times \mathbb{Z}_T$ and the static category **Ham** with $\mathbf{Ham}_{T\text{-per}}$; the mathematical content of Law III consists precisely in identifying the natural transformations of the resulting functor categories that correspond to genuinely non-equilibrium phases.

We reiterate: *the framework is modular, not unified.* Each law is a self-standing mathematical layer, and the emergent properties at each layer arise from the explicit composition functor stated here and in the corresponding sections of Laws III and IV.

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Code availability. A Haskell implementation accompanying this paper is available in the project repository under `src/phase-bound-matter`, providing types for phases, finite-group cohomology, and entanglement-entropy computations on small toric-code instances. The package compiles with `cabal build` and the test suite passes with `cabal test`. The companion paper Law I in the same modular series [19] provides the categorical primitives invoked throughout this paper.

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