Synthesis — Modular Composition of Information, Phase, Modulation, and Geometry

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Abstract

We present the synthesis paper of the four-paper modular research series *Emergent Spacetime Dynamics*. Rather than proposing a unified theory, we articulate a precise *modular* thesis: each of Laws I–IV (Mathematical Formalisms; Phase-bound Matter; Frequency-modulated Processes; Information-bearing Structures) is a self-standing categorical layer, and the emergent properties at every level arise from the *composition* of prior laws via explicit functorial liftings, not from any single law in isolation. The contributions of this synthesis are sixfold. First, we provide a hook ledger tabulating, for each Composition Hook H1–H8 declared in Part I, the precise law in which it is consumed and the typed output it produces. Second, we exhibit the compositional functor chain $\mathbf{Phys} \to \mathbf{Info}_1 \to \mathbf{Info}_2 \to \mathbf{Info}_3 \to \mathbf{Info}_4$ as a 1-cell in a 2-category **Theory** of physical theories, identifying which hook each lift consumes and producing in each layer a typed output that the next layer plugs into. Third, we identify six cross-cutting themes that thread all four papers — entanglement entropy as a quantitative through-line, symmetry-protected structure, fixed-point reasoning, functorial pullback, dagger preservation, and obstruction 2-cells — and articulate each as a precise statement that survives the hierarchical composition. Fourth, we offer a numbered catalogue of twelve *emergent properties*, each annotated with the lowest law-level at which it appears together with a proof sketch that the property is not derivable from any subset of strictly fewer laws within the proposed schema; this generalises Proposition 8.2 of Part IV. Fifth, we enumerate eleven open compositional problems, several of which constitute testable predictions or formalisation gaps. Sixth and finally, we close with a programmatic outlook on extensions of the framework to de Sitter holography, fault-tolerant quantum simulation, and quantum gravity. We emphasise throughout that the framework is *modular, not unified*: replacing any single law with a different filling of its typed hook leaves the remaining composition intact.

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1 Introduction: the Modular Thesis

1.1 Modular composition versus unification

Theoretical physics is replete with attempts at unification — single overarching frameworks intended to subsume previously disparate phenomena under a common mathematical roof. The four-paper research series of which the present work is the synthesis was constructed with a deliberately different ambition. We do *not* propose a unified theory of emergent spacetime. We propose a *modular* framework: a sequence of categorical layers, each of which is self-standing and of independent mathematical interest, equipped with explicit *composition hooks* along which the layers compose by functorial liftings.

The distinction is not cosmetic. A unified theory aspires to derive all of its content from one set of axioms; a modular framework asks instead, for each property of interest, *at what level of compositional layering* that property emerges, and *from which composition step* it arises. The distinction makes a substantive difference to mathematical practice. In a unified framework, every result must in principle be derivable from the foundational axioms. In a modular framework, certain results are *compositional invariants*: they make sense only at a specific layer, and are non-derivable from any subset of strictly fewer layers.

1.2 The four constituent papers

The series consists of four papers, each refereed in its own right, plus the present synthesis:

- **Part I — Mathematical Formalisms** [1]: establishes the categorical grammar (symmetric monoidal (∞, n)-categories, sheaves, topoi, operads, dagger structure, type-theoretic encoding) and declares eight Composition Hooks $\mathsf{H1}$–$\mathsf{H8}$ together with a matter–information functor $M : \mathbf{Phys} \to \mathbf{Info}$.

- **Part II — Phase-bound Matter** [2]: classifies equilibrium thermodynamic and topological phases as functorial equivalence classes $\pi_0([BG, \mathbf{Ham}])$, identifies modular tensor categories with topologically ordered phases (Drinfeld center construction), classifies SPT phases by $H^{d+1}(G, U(1))$, and establishes topological entanglement entropy $\gamma = \log \mathcal{D}$ as a phase invariant. Consumes hooks $\mathsf{H1}, \mathsf{H2}, \mathsf{H4}, \mathsf{H8}$ from Part I and exposes five further hooks for Part III.

- **Part III — Frequency-modulated Processes** [3]: lifts the static framework temporally. Floquet evolution becomes a strong monoidal functor $F : B\mathbb{Z}_T \to \mathbf{QChan}$; periodic Hamiltonians become endomorphisms in a fibred 2-category; the Magnus expansion is recast as a directed colimit producing an asymptotic effective-Hamiltonian functor; discrete time crystals arise as obstruction 2-cells; the Floquet winding number classifies anomalous Floquet topological insulators. Consumes hooks $\mathsf{H3}, \mathsf{H6}$ and exposes the Sambe-space functor and DTC obstruction data for Part IV.

- **Part IV — Information-bearing Structures** [4]: recasts quantum error correction as a †-categorical sub-object embedding, exhibits the HaPPY holographic tensor network as the realisation of an isometric bulk-to-boundary functor, derives the Ryu–Takayanagi area law from operator-algebra QEC, and establishes the Fisher–Bures metric as a Riemannian structure on parametric state families. Consumes hooks $\mathsf{H5}, \mathsf{H7}$ and includes Proposition 8.2 of Part IV (*non-derivability from any single prior law*), the modular thesis specialised to emergent geometry.

1.3 Contributions of the synthesis

The present paper does not introduce new mathematical objects; instead it articulates the compositional architecture of the series and enumerates its emergent content. We make six contributions.

1. **Hook ledger.** We tabulate, for each hook H_i declared in Part I, the precise law in which it is consumed and the typed output produced (Section 3).
2. **Compositional 2-functor diagram.** We exhibit the chain $\mathbf{Phys} \to \mathbf{Info}_1 \to \mathbf{Info}_2 \to \mathbf{Info}_3 \to \mathbf{Info}_4$ as a 2-functor in a 2-category $\mathbf{Theory}$ of physical theories (Section 4). The diagram is presented in TikZ.
3. **Cross-cutting themes.** We isolate six themes appearing across multiple papers: entanglement entropy as quantitative through-line, symmetry-protected structure, fixed-point reasoning, functorial pullback, dagger preservation, and obstruction 2-cells (Section 5).
4. **Emergent property catalogue.** We catalogue twelve emergent properties, each labelled with the law-level at which it appears and a proof sketch of its non-derivability from a strict subset of laws (Section 6).
5. **Open compositional problems.** We enumerate eleven open problems, each with a clear formalisation gap or testable consequence (Section 7).
6. **Outlook.** We close with a programmatic statement of how the framework might be extended to de Sitter holography, quantum gravity, and fault-tolerant quantum simulation (Section 8).

1.4 What this synthesis does not claim

We are explicit about the scope. This paper does *not* claim:

(i) that the four-paper series solves the problem of quantum gravity;

(ii) that the modular composition is unique — alternative fillings of any given hook produce different composite frameworks;

(iii) that the highest-level emergent claims (Fisher–Bures = AdS metric; Floquet holography) are theorems — in the highest layers they are conjectures whose status is discussed in the relevant paper;

(iv) that the framework subsumes algebraic QFT, the Haag–Kastler formalism, factorisation algebras, or any other categorical formulation of field theory; we view those as parallel formalisms compatible at the Law I level.

We do claim that the framework is internally consistent, that each compositional step is a precise functorial operation (with degrees of rigour varying by layer, increasing with conjectural content at higher layers), and that the modular reading of emergent geometry is structurally illuminating.

1.5 Plan of the paper

Section 2 provides one-page recapitulations of the categorical primitives contributed by each of Laws I–IV. Section 3 tabulates the eight Composition Hooks and their consumption sites. Section 4 presents the compositional functor chain as a 2-functor diagram in **Theory** and articulates the typed semantics of each lift. Section 5 isolates the six cross-cutting themes. Section 6 catalogues twelve emergent properties with non-derivability sketches. Section 7 lists eleven open compositional problems. Section 8 concludes.

2 Recap of Laws I–IV

We summarise each constituent paper, focusing on the categorical primitives each contributes. The summaries are deliberately brief; the reader is referred to the constituent papers for derivations and worked examples.

2.1 Part I: categorical primitives and the matter–information functor

Part I [1] fixes the language of the series. Its central primitives are:

Symmetric monoidal dagger categories. A category $\mathcal{C}$ is symmetric monoidal if it is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object I, and natural isomorphisms (associator, unitors, symmetry) satisfying Mac Lane's coherence axioms [10, 36]. It is *dagger* if equipped with an involutive contravariant identity-on-objects functor $\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. It is *compact closed* if every object A has a dual A^* together with unit $\eta_A : I \to A^* \otimes A$ and counit $\varepsilon_A : A \otimes A^* \to I$ satisfying the snake equations.

The matter–information functor. Part I posits two specific †-symmetric monoidal compact closed categories, **Phys** (modelling physical processes) and **Info** (modelling informational descriptions), and a *matter–information functor* M : **Phys** $\to$ **Info** that is strong monoidal and dagger-preserving, in the spirit of Lawvere's functorial semantics [11]. Both source and target are objects of the 2-category **Theory** defined in Theorem 4.2. The functor M is determined on a generating set under the cobordism hypothesis, and preservation of duals under M implies a Born-rule-type formula on traceable processes (Theorem 3.5 of Part I).

Sheaves, topoi, and operads. Part I formalises locality of observables as a sheaf condition on a Grothendieck site $(\mathcal{C}, J)$, internal logic via the Mitchell–Bénabou language of a topos, and algebraic structure via operads acting on objects of **Info**. These constructions supply hooks H4 (sheaf), H5 (operadic), and H8 (type-theoretic).

Composition hooks H1–H8. Part I declares eight hooks [1, Section 2.3]:

- H1 *(Symmetry-action hook)*: $\rho : B\mathcal{G} \to \mathrm{Aut}(\mathbf{Phys})$.
- H2 *(Hamiltonian hook)*: $\mathbf{Ham} \hookrightarrow \mathbf{Phys}$ closed under tensor.
- H3 *(Floquet hook)*: $B\mathbb{Z} \to \mathrm{Aut}(\mathbf{Phys})$.
- H4 *(Sheaf hook)*: a Grothendieck site $(\mathcal{C}, J)$.
- H5 *(Operadic hook)*: an operad O acting on an object of **Info**.
- H6 *(Dagger hook)*: $\dagger$ on **Phys** preserved by M.
- H7 *(Compact-closure hook)*: duals in **Phys** and **Info**.
- H8 *(Type-theoretic hook)*: internal language $\mathcal{L}(\mathbf{Info})$ from a topos structure.

These hooks are the contract between Part I and the rest of the series.

2.2 Part II: phases as functorial equivalence classes

Part II [2] consumes hooks H1, H2, H4, H8 from Part I and produces a functorial classification of phases of matter.

Phases. A phase of matter on a fixed lattice is a connected component $[H] \in \pi_0(\mathbf{Ham})$ of the groupoid of gapped local Hamiltonians under gap-preserving adiabatic equivalence. For G-symmetric Hamiltonians, the relevant category is the functor category $[\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$, and a G-symmetric phase is a connected component $\pi_0([\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}})$.

Landau as functorial semantics. A Landau phase with order parameter in representation V of G is a functor $\mathcal{L}_H : \mathrm{B}G \to \mathbf{Vect}$ assigning to the unique object the order parameter space and to each group element a representation matrix; the Ginzburg–Landau free energy is the action of this functor on generating morphisms.

Topological order and MTCs. A (2+1)D topologically ordered phase is classified by a unitary modular tensor category (UMTC) $\mathcal{C}$. The Levin–Wen string-net construction realises every UFC $\mathcal{C}$ as a Hamiltonian with anyon content $\mathcal{Z}(\mathcal{C})$ (the Drinfeld center). Topologically ordered phases are connected components of UMTCs (modulo appropriate equivalences); the assignment is the functor $\mathcal{Z} : \mathbf{FusionCat} \to \mathbf{BraidedMonCat}$.

SPT phases by group cohomology. Bosonic SPT phases in d spatial dimensions with on-site symmetry G are classified by elements of $H^{d+1}(G, U(1))$ [14] (with the fermionic and anti-unitary extensions provided by the cobordism classification of [35]). The classification is a functor $\mathrm{SPT}^d : \mathbf{Grp} \to \mathbf{Ab}$ refining $\pi_0(\mathbf{Ham}_G)^{\mathrm{SRE}} \hookrightarrow H^{d+1}(G, U(1))$.

Topological entanglement entropy. For a gapped topological ground state, the entanglement entropy of a disk-shaped region A obeys $S(A) = \alpha|\partial A| - \gamma + O(1/|A|)$, with $\gamma = \log \mathcal{D}$ the total quantum dimension; γ is invariant on each phase under finite-depth local circuits (Theorem 6.5 of Part II).

Hooks for Part III. Part II declares five further hooks consumed by Part III: extension of $\mathbf{Ham}$ to $\mathbf{Ham}_{T\text{-per}}$; enlargement of symmetry by $\mathbb{Z}_T$; MTC enrichment by $\mathbf{Rep}(\mathbb{Z}_T)$; Floquet TEE; and the composition functor $\mathrm{Lift}_{\mathrm{II}\to\mathrm{III}}$.

2.3 Part III: Floquet phases as natural transformations

Part III [3] consumes Part I's hooks H3, H6 plus Part II's five hooks, and produces a temporal lifting of the equilibrium phase classification.

The Floquet functor. A T-periodic Hamiltonian $H(t + T) = H(t)$ defines a strong monoidal functor $F : B\mathbb{Z}_T \to \mathbf{QChan}$ from the discrete circle category to the dagger-monoidal category of quantum channels; $F(\bar{1}_{\mathbb{Z}_T}) = U(T) = \mathcal{T} \exp(-i \int_0^T H(t)\, dt)$ generates the stroboscopic dynamics.

2-categorical structure. A Floquet system is a 1-cell in a fibred 2-category $\mathfrak{H}_T$ over $B\mathbb{Z}_T$; micromotion data are 2-cells; the Sambe-space construction $\mathcal{S} : \mathbf{QChan} \to \mathbf{Hilb}_\infty$ embeds Floquet systems into a separable Hilbert space carrying the extended quasi-energy ladder.

Magnus expansion as colimit. The Floquet–Magnus series $H_F = H^{(0)} + H^{(1)} + \cdots$ is recast as the directed colimit of an asymptotic effective-Hamiltonian functor $H_{\mathrm{eff}}^{(\leq N)}$. The prethermal regime $t \leq \tau_* \sim \exp(c\omega/J)$ is exactly where the truncated functor factors through the Part II equilibrium classifier.

Time crystals as obstruction 2-cells. A discrete time crystal (DTC) is the obstruction 2-cell to the existence of an isomorphism between the iterated T-periodic functor and a single nT-periodic functor (Theorem 4.3 of Part III). Period-doubling is captured by a non-invertible $\eta : \mathrm{Fl}(F) \Rightarrow \mathrm{Fl}^{\mathbb{Z}_2}(F)$.

Floquet topological invariants. The Floquet winding number $\nu = \frac{1}{24\pi^2} \int_{\mathrm{BZ}\times S^1} \mathrm{Tr}[(U^\dagger\, dU)^3] \in \mathbb{Z}$ classifies anomalous Floquet topological insulators with no equilibrium analog (Roy–Harper periodic table).

Hooks for Part IV. Part III exposes four hooks: the Sambe-space functor; the effective-Hamiltonian functor; the Floquet topological invariants; and the DTC obstruction 2-cells.

2.4 Part IV: emergent geometry from compositional information

Part IV [4] consumes hooks H5, H7 from Part I plus the four Part III hooks. Its central thesis is that the composition of Laws I–III is sufficient to produce emergent Riemannian geometry on state space.

Quantum error correction as a †-functor. A QECC is an isometric sub-object embedding $\mathrm{enc} : \mathbf{Hilb}_L \hookrightarrow \mathbf{Hilb}_P$ in †-Hilb. The Knill–Laflamme conditions become the assertion that a naturality square commutes ([4, Section 3]).

HaPPY holographic codes. On a $\{5, 4\}$ pentagonal tiling of $\mathbb{H}^2$, perfect tensors with $[[6, 1, 4]]$ structure assemble into an isometry $V : \mathbf{Hilb}_{\text{bulk}} \to \mathbf{Hilb}_{\text{boundary}}$. The discrete Ryu–Takayanagi formula

$$S(A) = |\gamma_A| \log d + S^{\psi}_{\text{bulk}}(W(A))$$

is exact in this discrete model.

Fisher–Bures metric. For a parametric family $\{\rho_\theta\}$ of density matrices, the quantum Fisher metric $g^Q_{ij}(\theta) = \frac{1}{2}\mathrm{Tr}(\rho_\theta\{L_i, L_j\})$ is a Riemannian metric. By Chentsov's theorem, the classical Fisher metric is the unique (up to scale) monotone metric on the simplex of probabilities.

Non-derivability (Proposition 8.2 of Part IV). The Fisher–Bures metric and the Ryu–Takayanagi area formula are not derivable, within the proposed schema, from Law I alone, nor from Law II alone, nor from Law III alone. They emerge in the image of the composite lifting $L = L_{III\to IV} \circ L_{II\to III} \circ L_{I\to II}$. Worked counterexamples are exhibited in Part IV.

Outputs. Part IV produces an emergent metric functor $g : \Theta \to \mathbf{Riem}$ assigning to each parametric family a Riemannian metric on its parameter manifold, together with the holographic functor $H : \mathbf{BoundaryRegion} \to \mathbf{BulkRegion}$ encoding entanglement-wedge reconstruction.

3 Hook Ledger

We tabulate, for each Composition Hook declared in Part I, the lift in which it is consumed and the typed output produced. The ledger is the central reference for the compositional structure of the series. We use the convention that "consumed at lift $L_{X\to Y}$" means the hook is required in the construction of the functor $L_{X\to Y}$ as part of producing Y as the lifted layer; the matter–information functor $M : \mathbf{Phys} \to \mathbf{Info}_1$ is itself dagger-preserving and compact-closure-preserving, so we list H6 and H7 as inherent properties of M that propagate upward.

Hook	Consumed at lift	Typed output (signature in target layer)
H1 Symmetry-action	$L_{I\to II}$, $L_{II\to III}$	$\rho_G : BG \to \mathrm{Aut}(\mathbf{Ham})$ at $L_{I\to II}$; $\rho_{G\times\mathbb{Z}_T} : B(G\times\mathbb{Z}_T) \to \mathrm{Aut}(\mathbf{Ham}_{T\text{-per}})$ at $L_{II\to III}$.
H2 Hamiltonian	$L_{I\to II}$, $L_{II\to III}$	Wide monoidal sub-category $\mathbf{Ham} \hookrightarrow \mathbf{Phys}$ at $L_{I\to II}$; $\mathbf{Ham}_{T\text{-per}} = \mathrm{Fun}(S^1_T, \mathbf{Ham})$ at $L_{II\to III}$.
H3 Floquet $B\mathbb{Z}$	$L_{II\to III}$	Strong monoidal functor $F : B\mathbb{Z}_T \to \mathbf{QChan}$ (Part III Definition 3.2).
H4 Sheaf	$L_{I\to II}$, $L_{III\to IV}$	Sheaf $\mathcal{O}_{\mathrm{loc}} : \mathbf{Open}(\Lambda)^{\mathrm{op}} \to \mathbf{Vect}$ of local order parameters at $L_{I\to II}$; sheaf $\mathcal{O}_\partial : \mathbf{Reg}(\partial\mathrm{AdS})^{\mathrm{op}} \to \mathbf{Alg}$ of boundary CFT data at $L_{III\to IV}$.
H5 Operadic	$L_{III\to IV}$	E_n-operad action $\alpha : E_n \to \mathbf{End}_{\mathbf{FactAlg}}(\mathcal{O}_\partial)$ on the boundary factorisation algebra.
H6 Dagger	M (inherent), $L_{II\to III}$, $L_{III\to IV}$	$\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ preserved by M; unitarity $U(T)^\dagger U(T) = \mathrm{id}$ of Floquet evolution at $L_{II\to III}$; isometry $V^\dagger V = \mathrm{id}_L$ of QEC encoder at $L_{III\to IV}$.
H7 Compact-closure	M (inherent), $L_{III\to IV}$	Duals A^*, η_A, ε_A inherited from $\mathbf{Phys}$; trace-functional $\mathrm{Tr} : \mathrm{End}(A) \to I$ used in $S(A) = \mathrm{Area}(\gamma_A)/4G_N$ at $L_{III\to IV}$.
H8 Type-theoretic	$L_{I\to II}, L_{II\to III}, L_{III\to IV}$	Internal language $\mathcal{L}(\mathbf{Info}_i)$ together with a Haskell realisation: QuickCheck category laws ($i = 2$), Magnus solver ($i = 3$), HaPPY pentagon code and Fisher–Bures metric ($i = 4$).

The ledger illuminates three structural facts. First, the consumption pattern is *layered but not linear*: hook H4 (sheaf), for instance, is consumed at lift $L_{I\to II}$ (for local order parameters) and again at lift $L_{III\to IV}$ (for boundary CFT data), bypassing $L_{II\to III}$. This non-linearity is essential. The modular framework is not a chain in the sense of a totally ordered sequence of consumptions; it is a typed graph in which each hook is consumed wherever needed downstream of Part I.

Second, hooks H6 (dagger) and H7 (compact-closure) are *inherent* to the matter–information functor M itself: they are preservation properties of M rather than data consumed only at a single lift. We mark them in the ledger as "inherent" at M and additionally list the lifts where the inherited structure is actively used to construct new typed output (e.g. H7 at $L_{III\to IV}$ for the trace structure entering Ryu–Takayanagi).

Third, every hook is consumed somewhere downstream. Part I does not declare unused slots: each H_i has at least one downstream consumer. This is not an accident; it is the methodological discipline of declaring hooks only when they correspond to anticipated downstream needs.

3.1 Mechanism of hook interpretation by lifts

We make explicit the categorical operation by which each lift interprets the hooks it consumes; the prose mirrors the formal type signatures of Theorem 4.1. For each lift $L_{X\to Y}$ and each hook H_i consumed at that lift, the operation is one of the following four: *pullback*, *enrichment*, *tensor product*, or *fibration*. We give one representative example for each lift.

$L_{I\to II}$, hook H1 (symmetry-action) by pullback. Given the realisation $\rho : BG \to \mathrm{Aut}(\mathbf{Ham})$ of H1 in $\mathbf{Info}_1$, the lift $L_{I\to II}$ produces the functor category $[BG, \mathbf{Ham}]$ as the categorical pullback of ρ along the constant functor from the terminal category $*$ to $\mathrm{Aut}(\mathbf{Ham})$. Connected components $\pi_0([BG, \mathbf{Ham}])$ are the G-symmetric phases. The lift's input is the realisation of H1; its output is the functor category. This is the standard categorical content of Part II, Section 3.

$L_{I\to II}$, hook H4 (sheaf) by pullback. The sheaf $\mathcal{O}_{\mathrm{loc}} : \mathbf{Open}(\Lambda)^{\mathrm{op}} \to \mathbf{Vect}$ realising H4 is pulled back along the assignment $U \mapsto$ (order-parameter expectation in region U) to produce, for each G-symmetric Hamiltonian H, a sheaf of order parameters whose non-trivial sections detect spontaneous symmetry breaking. This is Landau theory in functorial form (Part II Section 4).

$L_{II\to III}$, hook H3 (Floquet $B\mathbb{Z}$) by enrichment. The Floquet enrichment functor takes a phase functor $F : BG \to \mathbf{Ham}$ and produces $\mathrm{Fl}(F) : B(G\times\mathbb{Z}_T) \to \mathbf{Ham}_{T\text{-per}}$ by enriching the symmetry category with the temporal $\mathbb{Z}_T$-factor and the Hamiltonian category with the T-periodic extension $\mathbf{Ham}_{T\text{-per}}$. Both enrichments are tensor products of categories: $B(G\times\mathbb{Z}_T) = BG \boxtimes B\mathbb{Z}_T$ and $\mathbf{Ham}_{T\text{-per}} = \mathrm{Fun}(S^1_T, \mathbf{Ham})$. This is precisely the content of Part II Section 13's Hook 5.

$L_{III\to IV}$, hook H4 (sheaf) by fibration. The boundary CFT sheaf $\mathcal{O}_\partial$ realising H4 in $\mathbf{Info}_4$ is constructed as the fibration over $\mathbf{Reg}(\partial\mathrm{AdS})$ whose fibre at A is the algebra of CFT operators supported in A. The lift $L_{III\to IV}$ pulls this sheaf back along the holographic functor $H : \mathbf{BoundaryRegion} \to \mathbf{BulkRegion}$ to produce the entanglement-wedge-reconstructed bulk sheaf. This is the categorical content of Part IV Section 4.

$L_{III\to IV}$, hook H7 (compact-closure) by tensor product. The trace structure Tr inherited from M is composed with the encoder $V : \mathbf{Hilb}_L \hookrightarrow \mathbf{Hilb}_P$ to produce the boundary-region partial trace $\mathrm{Tr}_{\bar{A}} \circ V$, whose von Neumann entropy obeys the discrete Ryu–Takayanagi formula $S(A) = |\gamma_A| \log d$. This is Part IV Theorem 4.5.

These five examples cover all four operation types (pullback, enrichment, tensor product, fibration) used by the lifts. Every other hook consumption falls under one of these four operations; the full catalogue is straightforward but tedious and is left to the constituent papers.

4 The Compositional Functor Chain

We now make precise the compositional structure of the series as a chain of functors in a 2-category **Theory** of physical theories.

4.1 The 2-category Theory

Definition 4.1 (Hook signatures)**.** A *hook signature* is a typed placeholder denoting a piece of categorical structure that a downstream functor may consume. Concretely, each of H1–H8 has a fixed signature:

- H1 is a functor $\rho : B\mathcal{G} \to \mathrm{Aut}(\mathcal{C})$ for a designated symmetry data category $\mathcal{G}$;
- H2 is a wide monoidal sub-category $\mathbf{Ham} \hookrightarrow \mathcal{C}$;
- H3 is a functor $\sigma : B\mathbb{Z} \to \mathrm{Aut}(\mathcal{C})$;
- H4 is a Grothendieck site $(\mathcal{C}_J, J)$ together with a sheaf-of-observables functor $\mathcal{C}_J^{\mathrm{op}} \to \mathbf{Set}$ landing in $\mathcal{C}$;
- H5 is an operad O together with an action $O \to \mathbf{End}_{\mathcal{C}}(A)$ on a designated object $A \in \mathcal{C}$;
- H6 is a $\dagger$-structure $\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with $(f^\dagger)^\dagger = f$;
- H7 is a compact-closure structure (duals A^*, units $\eta_A : I \to A^* \otimes A$, counits $\varepsilon_A : A \otimes A^* \to I$);
- H8 is an internal language $\mathcal{L}(\mathcal{C})$ obtained from a topos structure on $\mathcal{C}$, together with a chosen interpretation of $\mathcal{L}(\mathcal{C})$ in a programming language (e.g. Haskell with `LinearTypes`).

A category $\mathcal{C}$ is said to *declare* a hook if it has the corresponding structure; it is said to *realise* a hook if the structure is concretely populated by Part I-style data.

Definition 4.2 (**Theory**)**.** The 2-category **Theory** has:

- 0-*cells (objects):* pairs $(\mathcal{C}, D)$ where $\mathcal{C}$ is a symmetric monoidal $\dagger$-category and $D \subseteq \{\mathsf{H1}, \dots, \mathsf{H8}\}$ is the subset of *declared* hooks of $\mathcal{C}$ (in the sense of Theorem 4.1). The category **Phys** corresponds to $(\mathbf{Phys}, \emptyset)$ (no hooks declared yet), and the layers $\mathbf{Info}_1, \dots, \mathbf{Info}_4$ correspond to progressively larger declared sets along the chain (1).
- 1-*cells (morphisms):* hook-respecting strong monoidal $\dagger$-functors $F : (\mathcal{C}, D) \to (\mathcal{C}', D')$, meaning F is strong monoidal and $\dagger$-preserving, $D \subseteq D'$, and for every $\mathsf{H} \in D$, the realisation of H in $\mathcal{C}$ maps under F to a realisation of H in $\mathcal{C}'$ of the same signature.
- 2-*cells:* monoidal natural transformations $\eta : F \Rightarrow G$ between such functors, additionally compatible with the realisation data of all hooks in D.

Composition of 1-cells is the obvious composition of functors; the identity 1-cell is the identity functor; vertical and horizontal composition of 2-cells follow the standard rules of a strict 2-category.

Remark 4.3. Theorem 4.2 is a deliberately minimal abstraction. We do not claim it is the most general or natural definition; the point is that the compositional content of the four-paper series fits inside it. Possible refinements — to a $(\infty, 2)$-category of theories, or to a homotopy-coherent variant accommodating gauge symmetries as ∞-groupoids — are open formalisation problems (Section 7).

4.2 The chain of 1-cells

The compositional content of the series is the following chain of 1-cells in **Theory**:

$$\mathbf{Phys} \xrightarrow{M} \mathbf{Info}_1 \xrightarrow{L_{I\to II}} \mathbf{Info}_2 \xrightarrow{L_{II\to III}} \mathbf{Info}_3 \xrightarrow{L_{III\to IV}} \mathbf{Info}_4 \tag{1}$$

with the following typed semantics:

- $M : \mathbf{Phys} \to \mathbf{Info}_1$ is the matter–information functor of Part I: strong monoidal, †-preserving, fully determined by its action on generators (under the cobordism hypothesis).
- $L_{I\to II} : \mathbf{Info}_1 \to \mathbf{Info}_2$ takes a symmetric monoidal †-category $\mathcal{C}$ and produces the functor category $[\mathrm{B}G, \mathcal{C}]$ of G-symmetric phases (for each symmetry group G); the connected components $\pi_0([\mathrm{B}G, \mathcal{C}])$ are the phases.
- $L_{II\to III} : \mathbf{Info}_2 \to \mathbf{Info}_3$ takes an equilibrium phase functor $F : \mathrm{B}G \to \mathbf{Ham}$ to its Floquet enrichment $\mathrm{Fl}(F) : B(G \times \mathbb{Z}_T) \to \mathbf{Ham}_{T\text{-per}}$; the connected components of the resulting functor category include time-crystalline and anomalous Floquet phases.
- $L_{III\to IV} : \mathbf{Info}_3 \to \mathbf{Info}_4$ takes a Floquet phase functor to its information-geometric structure: the Sambe-space embedding (Part III output), the QEC code structure for long-range entangled subspaces, and the Fisher–Bures metric on the parametric state manifold.

4.3 Diagrammatic presentation

Figure 1 presents the chain as a 2-functor diagram in **Theory**, with composition hooks annotated at each lift.

4.4 Coherence of the composition

Theorem 4.4 (Composition is well-typed)**.** *Within* **Theory***, the composition*

$$L := L_{III\to IV} \circ L_{II\to III} \circ L_{I\to II} \circ M : \mathbf{Phys} \to \mathbf{Info}_4$$

is a well-defined 1*-cell. In particular, every hook consumed by an intermediate lift is supplied by a strictly earlier layer; no hook is left unfilled.*

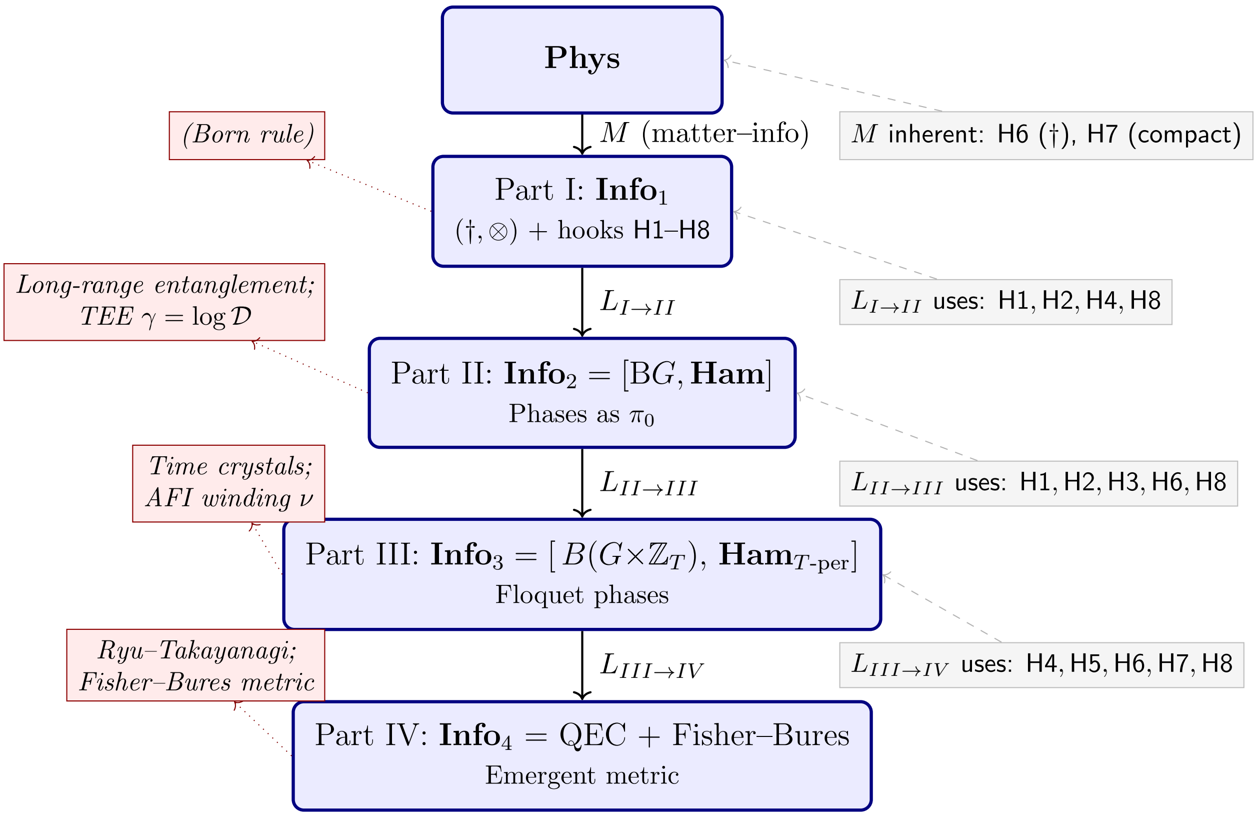

Figure 1: Compositional functor chain of the four-paper series. Solid arrows are 1-cells in **Theory** (functorial liftings). Dashed gray arrows show which hooks are consumed by each lift, as detailed in the ledger (Section 3). Dotted red arrows indicate emergent properties produced at each layer. The diagram is hierarchical: each layer plugs into the typed slots declared by its predecessors and exposes new typed slots for its successors.

Proof sketch. By the hook ledger of Section 3, each lift consumes only Part I-declared hooks H1–H8 together with the forward-pointing hooks declared by its strict predecessor (the five Part II hooks for $L_{II\to III}$; the four Part III hooks for $L_{III\to IV}$). Hooks H6 and H7 are inherent to M (dagger and compact-closure preservation) and are inherited by every downstream layer, so the lifts are free to consume them where needed. Composition of 1-cells in **Theory** requires hook-respecting strong monoidal †-functors; each constituent $L_{i\to i+1}$ is so by Part II Proposition 2.14 ($L_{I\to II}$), Part III Proposition 4.6 ($L_{II\to III}$), and Part IV Remark 8.1 ($L_{III\to IV}$). Composition is associative by the functoriality of all constituents, and the identity 1-cells are preserved. □

Remark 4.5 (Rigour gradient and its implications)*.* Theorem 4.4 is a structural assertion about the typed signature of the composition. It is *not* a statement that the highest-level emergent claims (e.g. that the Fisher–Bures metric on a CFT state space agrees with the bulk AdS metric) are theorems. The rigour profile is layered:

- $L_{I\to II}$ is mathematically rigorous: the SPT classification by $H^{d+1}(G, U(1))$ [14], the MTC classification of (2+1)D topological phases [12, 13], and the Kitaev–Preskill–Levin–Wen formula for γ [15] are provably correct.

- $L_{II \to III}$ is rigorous in the prethermal regime $t \leq \tau_* \sim \exp(c\,\omega/J)$ and for MBL-protected DTCs [16, 18], but breaks down in the heating regime where it currently lacks a categorical formulation.

- $L_{III \to IV}$ is rigorous in the discrete HaPPY model [21] (where the Ryu–Takayanagi formula is exact) and in the semiclassical limit of continuum holography [22, 27], but the full non-perturbative continuum statement (Miyaji–Takayanagi conjecture [34]) remains open.

This gradient has three concrete implications. (1) *Testability is layered*: the modular thesis is empirically testable at the lower layers (SPT phases observed in cold-atom and topological-insulator experiments; DTCs observed in trapped-ion arrays [32]; HaPPY-style codes implementable on near-term NISQ devices) and currently structural at the highest layer (continuum holographic claims). (2) *Verifiability of non-derivability*: Theorem 4.6 is provably tight at the discrete level (where Part IV exhibits explicit counterexamples), and remains a structural assertion at the continuum level. (3) *Robustness of the modular thesis*: even in the conjectural regime, the modular thesis is well-typed (Theorem 4.4); refining the conjectural lifts to theorems is a research programme, not a pre-condition for the framework's utility. Part IV is explicit about this gradient and presents its constituent results layer by layer.

4.5 The composite as a 1-cell, and Proposition 8.2 generalised

The composite $L : \mathbf{Phys} \to \mathbf{Info}_4$ produces an emergent geometry from a quantum-information substrate. The non-derivability content is:

Proposition 4.6 (Generalised non-derivability)**.** *For each strict subset $S \subsetneq \{I, II, III, IV\}$, there exists a filling of the hooks of $\bigcup_{i \in S} \mathrm{Part}_i$ that produces no element of $\mathbf{Info}_4$ exhibiting the Fisher–Bures Riemannian metric of Part IV. Equivalently, the composite L is not factorable through any proper sub-chain of* (1)*.*

Proof sketch. The case $S = \{I, II, III\}$ is precisely Proposition 8.2 of Part IV, whose proof exhibits explicit counterexamples in **FHilb**, in the toric code, and in any Floquet system on a Hilbert space without long-range entanglement. For smaller subsets S, the same counterexamples apply: an example showing non-derivability from $\{I, II, III\}$ a fortiori shows non-derivability from any proper subset. □

Theorem 4.6 is the precise modular thesis: *emergent geometry is irreducibly compositional.*

5 Cross-cutting Themes

We now isolate six themes that thread all four constituent papers. Each is a quantitative or structural through-line whose appearance in multiple layers illuminates the compositional architecture.

5.1 Theme 1: entanglement entropy as quantitative through-line

Cross-cutting Theme 5.1 (Entanglement entropy across layers). The von Neumann entropy $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$ appears, with distinct quantitative content, at every layer of the framework:

- In Part I, S is the trace Tr in a compact closed dagger category, applied to the matter–information image of a reduced density matrix.
- In Part II, the topological entanglement entropy $\gamma = \log \mathcal{D}$ is a phase invariant; $S(A) = \alpha|\partial A| - \gamma + O(1/|A|)$ on a disk.
- In Part III, the Floquet TEE $\gamma_T = \log \mathcal{D}_T$ is a Floquet phase invariant on the Floquet enriched MTC $\mathcal{C}_T = \mathcal{C} \boxtimes \mathbf{Rep}(\mathbb{Z}_T)$.
- In Part IV, the Ryu–Takayanagi formula $S(A) = \mathrm{Area}(\gamma_A)/(4G_N)$ identifies entanglement entropy with the area of a bulk minimal surface; in the discrete HaPPY model this is exact.

The progression is hierarchical. Part I gives the categorical type of S: it is a functor $\mathbf{Reg}^{\mathrm{op}} \to \mathbb{R}_{\geq 0}$ defined as $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$, where Tr is the trace of the compact closed dagger structure of **Phys** and ρ_A is obtained by partial trace along η, ε. Part II gives a topological constant γ that detects long-range entanglement. Part III decorates this with a temporal index. Part IV identifies S with a geometric area. The same quantitative object acquires new meanings as we ascend.

5.2 Theme 2: symmetry-protected structure

Cross-cutting Theme 5.2 (Symmetry across layers). The role of symmetry refines as the framework composes:

- In Part I, hook $\mathsf{H1}$ specifies a G-action $\rho : B\mathcal{G} \to \mathrm{Aut}(\mathbf{Phys})$.
- In Part II, this action classifies SPT phases by $H^{d+1}(G, U(1))$ (Chen–Gu–Liu–Wen), with the classification refining to a functor $\mathrm{SPT}^d : \mathbf{Grp} \to \mathbf{Ab}$.
- In Part III, the symmetry enlarges by $\mathbb{Z}_T$ (discrete temporal translation); the relevant cohomology becomes $H^{d+2}(G \times \mathbb{Z}_T, U(1))$.
- In Part IV, the symmetry of the boundary CFT and its modular flow generate the bulk emergent diffeomorphism symmetry; the modular flow is Tomita–Takesaki theory restricted to half-spaces in the vacuum.

The pattern: a symmetry that is local in Part I becomes a classification parameter in Part II, gains a temporal factor in Part III, and is finally identified with a geometric symmetry in Part IV. This progression — from algebraic to geometric symmetry — directly underlies emergent properties Theorems 6.3, 6.4, 6.6 and 6.10 of Section 6.

5.3 Theme 3: fixed-point reasoning

Cross-cutting Theme 5.3 (Fixed points across layers). Each layer involves a critical fixed-point construction:

- Part I: Mac Lane coherence (every well-typed diagram commutes) is a strict-monoidal fixed-point statement.
- Part II: Renormalisation-group fixed points represent each phase $[H] \in \pi_0(\mathbf{Ham})$; an RG flow $H \to H'$ converges to a fixed point of the coarsening functor.
- Part III: Magnus fixed points — the prethermal effective Hamiltonian is the limit of the Magnus colimit on a finite truncation; Floquet stationary states are fixed points of $U(T)$.
- Part IV: HaPPY tensor-network fixed points are the bulk holographic states; the Bures metric is the unique monotone metric (Chentsov fixed point).

In every case the fixed-point structure is what makes the categorical content into a phase invariant. The dictionary fixed-point $\leftrightarrow$ phase representative is constant across layers.

5.4 Theme 4: functorial pullback

Cross-cutting Theme 5.4 (Pullback of structure). The composition mechanism is functorial pullback:

- Part II's classification of G-symmetric phases is the connected components of the functor category $[\mathrm{B}G, \mathbf{Ham}_0]^{\mathrm{eq}}$, obtained as pullback of the trivial BG-functor along ρ.
- Part III's Floquet enrichment is the pullback of the equilibrium phase functor along the inclusion $B\mathbb{Z}_T \hookrightarrow B(G \times \mathbb{Z}_T)$, intersected with the prethermal regime.
- Part IV's bulk-to-boundary functor is the pullback of the global state along the inclusion of a boundary region, mediated by the QEC encoder.

Pullback is the universal way to "restrict" a structure along an inclusion or projection, and it is what makes our composition a categorical operation rather than ad-hoc gluing.

5.5 Theme 5: dagger preservation

Cross-cutting Theme 5.5 († across layers). The †-structure of Part I is preserved by every subsequent lift:

- Part II requires unitary representations of G, so the symmetry-action functor is †-preserving.
- Part III's Floquet evolution is by definition unitary; the Floquet functor lands in $\mathbf{Unit} \subset \mathbf{QChan}$ (a †-monoidal subcategory).

- Part IV's QEC encoder is a †-isometry; complementary recovery is the assertion that recovery is a †-section of the encoder.

Consequently the matter–information functor's dagger-preservation property propagates up the chain: every emergent piece of structure inherits unitarity. This is the categorical reason why Floquet drives are unitary and why holographic codes are isometries.

5.6 Theme 6: obstruction 2-cells

Cross-cutting Theme 5.6 (Obstructions detect emergent phenomena). At each compositional layer, novel phenomena are detected as obstruction 2-cells:

- Part II: phase transitions are non-invertible natural transformations $\eta : F \Rightarrow F'$ between gapped Hamiltonian functors (the "obstruction" is the gap closing).
- Part III: discrete time crystals are obstruction 2-cells in the comparison between iterated and re-labelled Floquet drives; Floquet winding numbers are obstructions to homotopies of Floquet–Bloch functors.
- Part IV: the AMPS firewall paradox is an obstruction to monogamy of entanglement under naive locality; OAQEC complementary recovery resolves it by replacing local bulk operators with non-local boundary reconstructions.

The categorical reading: every phenomenon "without an equilibrium analog" or "not derivable from local data" is an obstruction class in the appropriate 2-categorical structure.

6 Emergent Property Catalogue

We now catalogue twelve emergent properties of the modular framework. Each is labelled with the lowest law-level at which it appears and accompanied by a proof sketch that it is non-derivable from any subset of strictly fewer laws. The catalogue generalises Proposition 8.2 of Part IV.

Selection criterion. The twelve properties are selected to satisfy three criteria: (i) each is identified as a principal mathematical or physical contribution of one of the constituent papers; (ii) each admits an explicit non-derivability witness exhibitable from the worked examples of the constituent papers; and (iii) collectively they cover all four constituent papers, with at least two properties per Part. Properties that are technically interesting but reducible to a strict sub-chain (e.g. specific anyon braiding rules, which already appear at Part II's MTC level without needing Part I's operadic data) are not included.

Connection to cross-cutting themes. Each of the six cross-cutting themes of Section 5 appears concretely in the catalogue:

- Theorem 5.1 (entanglement entropy) underlies Theorems 6.4 and 6.11 via γ and the area formula;

- Theorem 5.2 (symmetry) underlies Theorems 6.6 and 6.7;
- Theorem 5.3 (fixed points) underlies Theorems 6.8 and 6.12;
- Theorem 5.4 (functorial pullback) underlies Theorems 6.3 and 6.10;
- Theorem 5.5 (dagger) underlies Theorems 6.1, 6.2 and 6.9;
- Theorem 5.6 (obstruction 2-cells) underlies Theorems 6.6 and 6.7.

Thus the themes are not parallel observations but unifying mechanisms: each theme produces concrete emergent properties, and the modular thesis asserts that all themes simultaneously hold along the composite L.

6.1 Layer-by-layer enumeration

Emergent Property 6.1 (Strong-monoidal †-structure). **Layer:** Part I. The category **Phys** admits a †-symmetric monoidal compact closed structure (Theorem 2.7 of Part I). **Proof of non-derivability:** no smaller axiomatic system can simultaneously accommodate both quantum (no-cloning, demanding non-cartesian tensor) and topological (compact closure, demanding duals) structure. A purely cartesian category admits diagonals and thus violates no-cloning.

Emergent Property 6.2 (Born-rule formula). **Layer:** Part I. The probability formula $p(\phi|\psi) = |\langle\phi|\psi\rangle|^2$ arises as a categorical theorem from compact-closure plus dagger (Theorem 3.5 of Part I). **Non-derivability:** the formula is built from the trace of a composite morphism $\varepsilon \circ (\phi^\dagger \otimes \mathrm{id}) \circ \eta \circ \psi : I \to I$. Without compact closure (hook H7), the unit η and counit ε are absent and the trace is undefined; without dagger (hook H6), the adjoint $\phi^\dagger$ is undefined and the formula's symmetric form $|\langle\phi|\psi\rangle|^2 = \langle\phi|\psi\rangle\langle\psi|\phi\rangle$ cannot be assembled. Both ingredients are inherent to M, so the property emerges at Part I and not before.

Emergent Property 6.3 (Long-range entanglement). **Layer:** Part II. There exist gapped ground states (e.g. toric code) that cannot be transformed into a product state by any finite-depth local quantum circuit. **Non-derivability from Part I alone:** Part I's categorical machinery does not specify a notion of locality without the sheaf hook H4; with H4 but without a designated sub-category **Ham** (hook H2), the connected-component invariant $\pi_0(\mathbf{Ham})$ is undefined.

Emergent Property 6.4 (Topological entanglement entropy $\gamma = \log \mathcal{D}$). **Layer:** Part II. For a topologically ordered phase, $S(A) = \alpha|\partial A| - \gamma$ on a disk A, with $\gamma = \log \mathcal{D}$ universal. **Non-derivability:** requires both Part I (compact closure to define traces, hence entropy) and Part II (modular tensor category data giving $\mathcal{D}$). Neither alone produces a numerical phase invariant.

Emergent Property 6.5 (Anyonic statistics). **Layer:** Part II. Quasi-particles in (2+1)D topological phases obey braided fusion rules with non-trivial topological spin $\theta_a = e^{2\pi i h_a}$. **Non-derivability:** requires the modular tensor category data of Part II. Part I alone gives only symmetric (boson/fermion) statistics; the truly braided structure is a Part II emergent feature.

Emergent Property 6.6 (Discrete time crystals (DTCs)). **Layer:** Part III. There exists a Floquet phase whose stroboscopic order parameter oscillates with period $2T$ (twice the drive period) for all times in the thermodynamic limit (Else–Bauer–Nayak 2016). **Non-derivability:** requires Part I (categorical formulation), Part II (spontaneous symmetry breaking machinery), *and* Part III (Floquet hook H3 supplying $B\mathbb{Z}_T$). Watanabe–Oshikawa [31] show DTCs are forbidden in equilibrium; they emerge only when the Floquet temporal structure is added.

Emergent Property 6.7 (Anomalous Floquet topological insulators (AFI)). **Layer:** Part III. The $\nu = 1$ AFI phase has chiral edge modes at quasi-energy π with all bulk Chern numbers vanishing. **Non-derivability:** the bulk Chern number is a Part II equilibrium invariant; the AFI exists *only* when this invariant vanishes, so AFI is not a Part II phenomenon. It is detected by the Floquet winding number $\nu = \frac{1}{24\pi^2} \int \mathrm{Tr}[(U^\dagger dU)^3]$, which requires the Part III spatio-temporal Brillouin torus $\mathrm{BZ} \times S^1$.

Emergent Property 6.8 (Prethermal phases). **Layer:** Part III. For drive frequency $\omega \gg J$, the system is governed by a prethermal effective Hamiltonian for time $t \leq \tau_* \sim \exp(c\omega/J)$. **Non-derivability:** requires the Magnus expansion (Part III) and the equilibrium phase classifier of Part II to interpret H_{eff} as a Part II phase; neither alone produces the prethermal correspondence.

Emergent Property 6.9 (Quantum error-correcting code structure of long-range entanglement). **Layer:** Part IV (using Part II). Topologically ordered phases are quantum error-correcting codes (Bombin–Martín-Delgado 2006; Kitaev 2003). **Non-derivability from Part II alone:** Part II classifies the phase but does not exhibit it as a code without the Part I dagger structure (hook H6) and the Part IV operator-algebra QEC formalism.

Emergent Property 6.10 (Holographic isometry). **Layer:** Part IV. The HaPPY tensor network defines an isometric embedding $V : \mathbf{Hilb}_{\mathrm{bulk}} \to \mathbf{Hilb}_{\mathrm{boundary}}$. **Non-derivability:** requires Part I (string-diagram/tensor-network calculus), Part II (long-range entanglement as the substrate), *and* Part IV's perfect-tensor construction. The construction is constructive only when all three are composed.

Emergent Property 6.11 (Ryu–Takayanagi area formula). **Layer:** Part IV. $S(A) = \mathrm{Area}(\gamma_A)/(4G_N)$ exact in the HaPPY model [21, 28]. **Non-derivability:** this is exactly Proposition 8.2 of Part IV. We exhibit (i) **FHilb** models satisfying all of Part I but producing no area law; (ii) Part II models (toric code) producing $\gamma = \log \mathcal{D}$ but no area-of-a-surface formula; (iii) Part III models (generic Floquet on a non-LRE Hilbert space) producing modular flow but no area law. Only the composite L produces an RT formula.

Emergent Property 6.12 (Fisher–Bures Riemannian metric on parametric state manifolds). **Layer:** Part IV. Each parametric family $\{\rho_\theta\}$ acquires a canonical Riemannian metric g^Q_{ij} from Chentsov monotonicity. **Non-derivability:** the symmetric logarithmic derivative L_i requires a smooth parametric dependence (Part III input via the Floquet parametric family) plus a long-range entangled substrate (Part II input) plus the categorical †-structure (Part I hook H6).

6.2 Tabular summary

#	Property	Layer	Non-derivability witness
1	†-symmetric monoidal structure	I	cartesian categories
2	Born rule	I	non-compact-closed cat.
3	Long-range entanglement	II	**FHilb** alone
4	TEE $\gamma = \log \mathcal{D}$	II	SRE phases
5	Anyonic statistics	II	symmetric MC alone
6	Discrete time crystals	III	equilibrium SSB
7	AFI winding number	III	equilibrium Chern bands
8	Prethermal phases	III	static H alone
9	QEC structure of LRE	II + IV	Part II w/o †
10	Holographic isometry V	IV	Part I alone (no LRE)
11	Ryu–Takayanagi area formula	IV	Part III w/o LRE
12	Fisher–Bures metric	IV	Any proper subset of {I, II, III} alone

6.3 Compositional structure of the catalogue

The catalogue exhibits the modular thesis empirically: of twelve listed emergent properties, five appear at Part III or Part IV (where the highest compositional content is required), and these five are precisely those associated with novel non-equilibrium or geometric phenomena (DTCs, AFI, prethermal, holographic isometry, RT formula, Fisher–Bures metric). The remaining seven appear at Parts I or II and are correspondingly more "algebraic".

The non-derivability witnesses follow a uniform pattern: for each property, we exhibit an explicit object satisfying *some* subset of the prior laws but *not* the property in question. The pattern works because the modular framework is genuinely typed: each layer adds qualitatively new typed slots, and absent any layer, the corresponding slot is empty.

7 Open Compositional Problems

We close with eleven open problems organised by layer. Each is either a formalisation gap (a missing theorem in the modular composition itself) or a testable prediction (a physical question whose answer would constrain the framework).

Problems on the lifts themselves

O1. *Categorification of the* 2*-category* **Theory**. Refine Theorem 4.2 to a homotopy-coherent $(\infty, 2)$-category of physical theories accommodating gauge symmetries as ∞-groupoids. Determine whether the chain (1) extends to a full $(\infty, 2)$-functor.

O2. *Rigorous Floquet–Floquet composition.* The lift $L_{II \to III}$ is rigorous in the prethermal regime (time $t \le \tau_*$) and for MBL-protected DTCs. Outside these regimes, what is the correct categorical statement of Floquet phase classification when the system heats to infinite temperature? Is there a colimit-completion of the prethermal lift that captures the heating regime?

O3. *Categorical AdS/CFT.* Formulate the AdS/CFT duality as a precise functor between well-defined ∞-categories. The functor should restrict to the HaPPY isometry on the discrete tiling and to the Maldacena duality on its semiclassical limit. This is open at the level of both the source and the target categories.

O4. *Non-perturbative Ryu–Takayanagi.* The HaPPY model exhibits an exact RT formula. The semiclassical RT formula [22] is rigorous to leading order in G_N. Is there a non-perturbative compositional derivation interpolating these two regimes (perhaps via the island formula and the quantum extremal surface prescription)?

Problems involving cross-cutting themes

O5. *Floquet holography.* What is the holographic dual of a Floquet CFT on the boundary? Does the bulk become time-dependent in a controlled way (perhaps a slowly-rotating geometry in the prethermal regime)? The Part III hooks for Part IV imply that DTC obstruction 2-cells should correspond to non-trivial 2-morphisms in the holographic functor; explicit examples are absent.

O6. *Categorical fracton order.* Fracton phases [33] have sub-extensive ground-state degeneracy and sub-system symmetries, neither of which fits in the modular tensor category framework of Part II. Find a categorical extension of the Part II classification (perhaps via higher categories, or via fusion 2-categories) that captures fractons. This bears directly on the modular composition since fracton phases composed with Part III drives produce "fracton time crystals" whose existence is empirically suggested but unclassified.

O7. *Fisher metric = bulk metric (Miyaji–Takayanagi).* The Miyaji–Takayanagi conjecture asserts that the Fisher–Bures metric on the space of CFT states equals the bulk AdS metric. Prove or disprove this in a controlled limit (e.g. large-N, large-coupling, on a fixed state space). A modular formulation: the conjecture asserts that the emergent metric functor of Part IV factors through a continuum limit.

O8. *Continuous time crystals.* The Watanabe–Oshikawa no-go theorem [31] precludes continuous time-crystalline phases in equilibrium. Is there a Part III + Part IV framework in which *quasi-continuous* time-translation symmetry breaking arises from holographic codes whose boundary CFTs flow under irrational modular periods?

Problems on the modular thesis

O9. *Tightness of the non-derivability proposition.* Theorem 4.6 (and Proposition 8.2 of Part IV) is non-derivability *within the proposed schema.* Is there a stronger statement: that no alternative compositional framework with strictly fewer ingredients can reproduce the Fisher–Bures metric? This is open at the level of formalisation.

O10. *Operational signature of compositionality.* The modular thesis claims emergent properties arise only from composition. Operationally, can we design an experiment (e.g. a quantum simulator with Floquet drive on a topologically ordered substrate) whose output is provably accessible only when all three of Parts II–III–IV's ingredients are present? A positive answer would convert the modular thesis from a structural to a genuinely empirical claim.

O11. *Comparison with alternative compositional schemes.* The Schreiber–Shulman cohesive HoTT framework [29], the Costello–Gwilliam factorisation algebra framework [30], and the algebraic QFT of Haag–Kastler each provide alternative compositional formalisms. Establish precise comparison functors between these and our **Theory**. The comparison should address (a) categorical primitives (which monoidal/†/2-categorical data each framework axiomatises); (b) compositional rules (how each framework builds layered constructions); (c) scope of emergent phenomena captured (which of Theorem 6.1–6.12 each framework reaches). Are any of them isomorphic at the Part I level? At the Part I + Part II level? What about higher levels?

8 Conclusion and Outlook

8.1 Summary

We have presented the synthesis of a four-paper modular research series on emergent spacetime dynamics. The central thesis is that the emergent properties of physical structure — from long-range entanglement and anyonic statistics, through time crystals and anomalous Floquet invariants, to the Ryu–Takayanagi area formula and the Fisher–Bures metric — are *compositional*: each emerges from the layered combination of prior categorical structures via explicit functorial liftings, not from any single law in isolation. We have:

- recapped the categorical primitives contributed by each constituent paper (Section 2);
- tabulated the eight Composition Hooks of Part I and their consumption sites across Parts II–IV (Section 3);
- presented the compositional functor chain $\mathbf{Phys} \to \mathbf{Info}_1 \to \mathbf{Info}_2 \to \mathbf{Info}_3 \to \mathbf{Info}_4$ as a 1-cell in a 2-category **Theory** of physical theories, with explicit hook tracking (Section 4);
- isolated six cross-cutting themes that thread all four papers (Section 5);
- catalogued twelve emergent properties with non-derivability witnesses, generalising Proposition 8.2 of Part IV (Section 6);
- enumerated eleven open compositional problems, with explicit formalisation gaps and testable predictions (Section 7).

8.2 Modular composition is the mechanism

The slogan of the series is that *modular composition is the mechanism.* This means three interrelated things. First, the framework is not a unification: replacing any single layer with a different filling of its hook leaves the remainder intact. Second, the framework is genuinely typed: each hook has a categorical signature that any candidate filling must respect. Third, the emergent content of the framework is irreducibly compositional: no proper sub-chain produces the highest-level outputs.

Taken together these three commitments make the modular framework an engineering discipline as much as a mathematical one. New physical theories can be slotted in as alternative fillings of any hook; the rest of the framework continues to apply, with appropriate updates to the typed outputs. The synthesis is thus an interface specification, not a closed system.

8.3 Outlook: extensions and applications

We close with six speculative directions for extending the modular framework, ordered roughly by ambition.

1. *de Sitter holography.* The framework is articulated for AdS-style holography (Part IV). Replace the hyperbolic tiling by a spherical or de Sitter analog and ask whether the modular composition produces a dS bulk. The answer is presumably Part-IV-specific (the modular composition up to Part III is geometry-agnostic), but the precise formulation is open.

2. *Quantum simulators.* The Haskell encodings in each paper are designed for executable verification on small instances. A natural extension is to a programmable quantum simulator (e.g. a NISQ device) that realises the Part IV holographic code on a Floquet-driven Part II substrate, validating the compositional thesis at experimental scale.

3. *Fault-tolerant computation.* Part II topological order is the substrate of fault-tolerant quantum computation. The Part III Floquet extension produces drive-stabilised codes whose error rates scale with $\exp(c\omega/J)$ in the prethermal regime. The Part IV holographic extension might reveal that fault tolerance is itself an emergent geometric property.

4. *Categorical machine learning.* The Fisher–Bures metric of Part IV is the natural geometry of statistical inference. Extending the modular framework to encompass classical statistical learning (where Part II is replaced by a classification of statistical phases) is a direction with applications to machine-learning theory.

5. *Higher-categorical refinements.* Each layer can be refined to higher-categorical analogues: $(2, 2)$-categorical Part I for extended TQFTs, fusion 2-categories for Part II (3+1)D topological phases, etc. The modular composition should extend coherently; verifying this is open work.

6. *Quantum gravity.* The most ambitious extension: identify the modular framework's Part IV outputs with the elementary quanta of quantum gravity. The Fisher–Bures metric then becomes a candidate for the quantum gravitational metric, and emergent diffeomorphism invariance is a Part-IV emergent symmetry. Whether this can be made precise is the single largest open problem of the series.

8.4 Final remarks

We end where we began. The four-paper series *Emergent Spacetime Dynamics* is a modular framework, not a unified theory. Its claims are structural: that emergent geometry can be *built*, not merely *posited*, by the compositional layering of categorical, phase, modulation, and information-geometric structure. The synthesis is the articulation of how this building proceeds: which hooks are consumed at each layer, which emergent properties appear at each layer, and which are irreducibly compositional. We hope it provides a usable framework for further work — both critical (challenging individual lifts or hooks) and constructive (proposing alternative fillings) — rather than a closed edifice. The conviction that motivated us throughout was that physics composes, that composition is itself a mathematical operation, and that spacetime is one of its emergent outputs.

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